Preconditioning linear systems of the Navier–Stokes equations

Ahmed Sameh*, Sunil Sathe, Keith Stein and Tayfun E. Tezduyar

*Department of Computer Science, Purdue University
250 N. University Street, West Lafayette, IN 47907, USA
sameh@cs.purdue.edu

Abstract
In this paper we present an effective preconditioning strategy for solving nonsymmetric linear systems that arise from the stabilized finite element formulations for incompressible fluid flow computations. These linear systems are solved via preconditioned Krylov subspace methods in which linear systems involving the preconditioner are solved via a Richardson scheme. We analyze this nested iterative scheme and provide numerical experiments to demonstrate its robustness and scalability.

1 A nested iterative scheme

In the stabilized finite element formulations for incompressible flow computations, one needs to solve linear systems of equations of the form, Tezduyar [1]

\[
\begin{bmatrix}
A & B \\
K^T & -C
\end{bmatrix}
\begin{bmatrix}
u \\
p
\end{bmatrix}
=
\begin{bmatrix}
a \\
b
\end{bmatrix},
\]  

in the inner-most loop of the computational process. Here, \(A \in \mathbb{R}^{n \times n}\) and \(B, K \in \mathbb{R}^{n \times m}\) with \(m \leq n\), and the coefficient matrix

\[
\mathcal{A} = \begin{bmatrix}
A & B \\
K^T & -C
\end{bmatrix},
\]

is nonsingular. In this paper, \(A\) is assumed to be nonsymmetric positive-stable, and \(B, K\) are of full column rank. We solve (1) via a preconditioned Krylov subspace method, such as GMRES, with a structured preconditioner \(\mathcal{M}\) given by, Golub and Wathen [2]

\[
\mathcal{M} = \frac{1}{2} (A + A^T) = \begin{bmatrix}
A_s & N \\
N^T & -C_s
\end{bmatrix},
\]

where \(A_s\) and \(C_s\) are the symmetric parts of \(A\), and \(C\), respectively. Throughout, we assume that \(\|A_s\| \geq \|A_{ss}\|\) in the Frobenius norm, where \(A_{ss} = (A - A^T)/2\) is the skew symmetric part of \(A\). The application of the preconditioner \(\mathcal{M}\) in each Krylov iteration requires the solution of linear systems of the form
For large-scale simulations, solving linear systems involving iteration, it is necessary to solve the systems in the inner iteration with relatively high accuracy. Similar to the classical Uzawa scheme. It turns out that in order to ensure convergence of the outer iteration, it is necessary to solve the systems in the inner iteration with relatively high accuracy.

Using a conjugate gradient algorithm for solving (3b) and (3a), we create an inner-outer iterative scheme similar to the classical Uzawa scheme. It turns out that in order to ensure convergence of the outer iteration, it is necessary to solve the systems in the inner iteration with relatively high accuracy.

For large-scale simulations, solving linear systems involving $A_s$ or $(C_s + N^T A_s^{-1} N)$ is not practical, as the action of $A_s^{-1}$ must be computed on various vectors. Consequently, the approach we adopt here is to replace the cost of computing the action of $A_s^{-1}$ by the cost of evaluating the action of some other “more economical” symmetric positive definite operator $\hat{A}^{-1}$ which approximates $A_s^{-1}$ in some sense. Thus, the linear system (3a) may be solved via the iteration

$$ x_{k+1} = (I - \hat{A}^{-1} A_s) x_k + \hat{A}^{-1} \tilde{f}, $$

where $\tilde{f} = f - N y$ and $\hat{A}$ is an appropriate symmetric positive definite splitting that assures convergence, i.e., $\alpha = \rho(I - \hat{A}^{-1} A_s) < 1$, where $\rho(\cdot)$ is the spectral radius. Similarly, we replace $A_s$ by $\hat{A}$ in (3b) and solve the resulting “inexact” system,

$$ (C_s + N^T \hat{A}^{-1} N) y = N^T \hat{A}^{-1} f - g, $$

instead of the original system (3b), via the iteration

$$ y_{k+1} = \left[ I - \hat{G}^{-1}(N^T \hat{A}^{-1} N + C_s) \right] y_k + \hat{G}^{-1} \tilde{s} $$

where $\tilde{s} = N^T \hat{A}^{-1} f - g$, and $\hat{G}^{-1}$ is an inexpensive symmetric positive definite approximation of the inverse of the inexact Schur complement $(N^T \hat{A}^{-1} N + C_s)$ that assures convergence of (4), i.e., $\beta = \rho(I - \hat{G}^{-1}(N^T \hat{A}^{-1} N + C_s)) < 1$. Moreover, $\hat{G}^{-1}$ is chosen such that $(I - \hat{G}^{-1/2}(N^T \hat{A}^{-1} N + C_s)\hat{G}^{-1/2})$ is positive definite.

Similarly, if we define the symmetric preconditioner $\hat{M}$ to the system (3) as

$$ \hat{M} = \left[ \begin{array}{c|c} A_s & N \\ \hline N^T & -C_s \end{array} \right], $$

we obtain the following preconditioned Richardson iterative scheme for solving (3), Baggag and Sameh [3]

$$ \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \hat{M}^{-1} \left[ \begin{array}{c} f \\ B^T -C_s \end{array} \right] \begin{bmatrix} x_k \\ y_k \end{bmatrix}, $$

that is convergent if and only if $\rho(I - \hat{M}^{-1} \hat{M}) < 1$.

Thus, our proposed nested iterative scheme is given as shown in Figure 1 in which the outermost iteration is that of a Krylov subspace method (we use GMRES throughout this paper), and the preconditioning operation is that of the preconditioned Richardson iteration (5).

Further, we analyze the iterative scheme (5) and give sufficient conditions for the monotone convergence of this inner iteration, and relate its rate of convergence to the outer iterations even though (4) is not the iteration that corresponds to the exact system (3b), but to a modified one (3d). In addition, we show that (4) need not be solved accurately. Numerical experiments are provided to compare the performance of our linear system solver with other direct and preconditioned iterative schemes.
Solve \[ \begin{bmatrix} A & B \\ K^T - C \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} ; \quad A \neq A^T \]

via a Krylov subspace method.

Preconditioner: \[ \mathcal{M} = \begin{bmatrix} A_s & N \\ N^T & -C_s \end{bmatrix} \]

\[ A_s = (A + A^T)/2 \]

Solve \[ \mathcal{M}z = r \]

Use the preconditioned Richardson iteration

\[ z_{k+1} = z_k + \hat{\mathcal{M}}^{-1} (r - \mathcal{M}z_k) \]

where \[ \hat{\mathcal{M}} = \begin{bmatrix} \hat{A} & N \\ N^T - \hat{G} + (N^T \hat{A}^{-1} N + C_s) \end{bmatrix} \]

Figure 1: A nested iterative scheme.

References

