1 Introduction

In recent decades, we have seen a substantial interest in and emphasis on using stabilized formulations in flow simulation and modeling with the finite element method. Streamline-upwind/Petrov-Galerkin (SUPG) formulation for incompressible flows, [1], SUPG formulation for compressible flows, [2], Galerkin/least-squares (GLS) formulation, [3], and pressure-stabilizing/Petrov-Galerkin (PSPG) formulation for incompressible flows, [4] are some of the most significant stabilized formulations that found usage in a wide range of applications. Many real-world flow problems are included among the applications that were addressed. These stabilized formulations became so attractive primarily because they stabilize the method without introducing excessive numerical dissipation. It is in this mindful way that they prevent numerical oscillations and other instabilities in solving problems with high Reynolds and/or Mach numbers and shocks and strong boundary layers, as well as when using equal-order interpolation functions for velocity and pressure and other unknowns. It was pointed out in [5] that these stabilized formulations also substantially improve the convergence rate in iterative solution of the large, matrix systems. Such matrix systems are solved at every Newton-Raphson step in iterative solution of the coupled nonlinear equation systems generated at every time level of a simulation.

The SUPG, GLS and PSPG formulations all include a stabilization parameter that is mostly referred to in the literature as \( \tau \). In general, this parameter might involve a measure of the local length scale (i.e., the "element length") and other factors such as the local Reynolds and Courant numbers. Various element lengths and \( \tau \) were proposed for the SUPG formulation, starting with those proposed in [6] and [2], and followed by the one introduced in [7]. More element lengths and \( \tau \) were prescribed for the SUPG, GLS, and PSPG methods reported later. Some other \( \tau \)s, dependent upon spatial and temporal discretizations, were introduced and tested in [8]. Later, \( \tau \)s which are applicable to higher-order elements were proposed in [9].

Recently, new ways of computing the \( \tau \)s based on the element-level matrices and vectors were introduced in [10]. These new definitions are expressed in terms of the ratios of the norms of the relevant matrices or vectors. They automatically take into account the local scales, advection field, and the element-level Reynolds number. Based on these definitions, a \( \tau \) can be calculated for each element, or even for each element node or degree-of-freedom or element equation. It was also shown in [10] that these \( \tau \)s, when calculated for each element, yield values quite comparable to those calculated based on the definition introduced in [7]. In conjunction with these stabilization parameters, in [11], a discontinuity-capturing directional dissipation stabilization was introduced as a potential alternative or complement to the LSIC (least-squares on incompressibility constraint) stabilization. A second element length scale based on the solution gradient was also introduced in [11]. This new element length scale would be used together with the element length scales already defined (directly or indirectly) in [10]. New stabilization parameters for the diffusive limit were introduced in [12]. These new parameters are closely related to the second element length scale that was introduced in [11]. That second element length scale can be recognized in [12] as a diffusion length scale.

In this paper we carry out a comparative investigation of the stabilization parameters and element length scales defined in the above references, as well as the element length scales defined in other work (see [6,13]). These comparisons include extensions of all these stabilization parameters and element length scales to higher-order elements. Furthermore, we compare the numerical viscosities generated by the SUPG stabilization with the eddy viscosity introduced by a Smagorinsky turbulence model, [14], specifically one that is based on element length scales, [15].

2 Formulations and Stabilization Parameters

2.1 Advection-Diffusion Equation. Consider over a domain \( \Omega \) with boundary \( \Gamma \) the following time-dependent advection-diffusion equation, written on \( \Omega \) and \( \forall t \in (0,T) \) as

\[
\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - \nabla \cdot (\nu \nabla \phi) = 0, \tag{1}
\]

where \( \phi \) represents the transported quantity, \( u \) is a divergence-free advection field, and \( \nu \) is the diffusivity. The essential and natural boundary conditions associated with Eq. (1) are

\[
\phi = g \quad \text{on} \quad \Gamma_{\text{c}}, \quad \nu \nabla \phi = h \quad \text{on} \quad \Gamma_{\text{n}}, \tag{2}
\]

where \( \Gamma_c \) and \( \Gamma_n \) are complementary subsets of the boundary \( \Gamma \); \( \nu \) is the unit normal vector, and \( g \) and \( h \) are given functions. A function \( \psi_{\phi}(x) \) is specified as the initial condition.

Given suitably defined finite-dimensional trial solution and test function spaces \( S_{\phi}^{lb} \) and \( V_{\phi}^{rb} \), the stabilized finite element formulation of Eq. (1) can be written as follows: find \( \phi \in S_{\phi}^{lb} \) such that

\[
\nabla \psi_{\phi}^{rb} \in V_{\phi}^{rb}.
\]

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\[
\begin{align*}
\int_{\Omega} w^h \left( \frac{\partial \phi^h}{\partial t} + u^h \cdot \nabla \phi^h \right) d\Omega + \int_{\Omega} \nabla w^h \cdot \nu \nabla \phi^h d\Omega - \int_{\Gamma_h} u^h \cdot \nu d\Gamma &= 0, \\
+ \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\text{SUPG}} u^h \cdot \nabla \phi^h \left( \frac{\partial \phi^h}{\partial t} + u^h \cdot \nabla \phi^h - \nu \nabla \cdot \nabla \phi^h \right) d\Omega &= 0,
\end{align*}
\]

Here \( n_{el} \) is the number of elements, \( \Omega^e \) is the domain for element \( e \), and \( \tau_{\text{SUPG}} \) is the SUPG stabilization parameter.

With the notation \( b \cdot f_{\Omega^e}( \ldots ) d\Omega \cdot b_V \) denoting the element-level matrix \( b \) and element-level vector \( b_V \) corresponding to the element-level integral \( \int_{\Omega^e}( \ldots ) d\Omega \). The element-level matrices and vectors are defined as follows:

\[
\begin{align*}
m: & \quad \int_{\Omega^e} w^h \frac{\partial \phi^h}{\partial t} d\Omega: m_V, \\
c: & \quad \int_{\Omega^e} w^h u^h \cdot \nabla \phi^h d\Omega: c_V, \\
k: & \quad \int_{\Omega^e} \nabla w^h \cdot \nu \nabla \phi^h d\Omega: k_V, \\
\tilde{k}: & \quad \int_{\Omega^e} u^h \cdot \nabla w^h u^h \cdot \nabla \phi^h d\Omega: \tilde{k}_V, \\
\tilde{c}: & \quad \int_{\Omega^e} u^h \cdot \nabla w^h \frac{\partial \phi^h}{\partial t} d\Omega: \tilde{c}_V.
\end{align*}
\]

From [10], the element-level Reynolds and Courant numbers can be written as

\[
\begin{align*}
\text{Re} &= \frac{u^h L^2}{\nu}, \\
\text{Cr}_u &= \frac{\Delta t}{2} \frac{||c||}{||m||}, \\
\text{Cr}_p &= \frac{\Delta t}{2} \frac{||e||}{||k||}, \\
\text{Cr}_p &= \frac{\Delta t}{2} \frac{||e||}{||k||}.
\end{align*}
\]

where \( ||b|| \) is the norm of matrix \( b \). Also from [10], we write the components of the element-matrix-based \( \tau_{\text{SUPG}} \):

\[
\begin{align*}
\tau_{S1} &= \frac{||e||}{||k||}, \\
\tau_{S2} &= \frac{\Delta t}{2} \frac{||e||}{||m||}, \\
\tau_{S3} &= \tau_{S1} \text{Re} = \frac{||e||}{||k||} \text{Re},
\end{align*}
\]

and the construction of \( \tau_{\text{SUPG}} \):

\[
\tau_{\text{SUPG}} = \left( \frac{1}{\tau_{S1}} + \frac{1}{\tau_{S2}} + \frac{1}{\tau_{S3}} \right)^{-1}.\quad (16)
\]

We note that \( \tau_{S1}, \tau_{S2}, \) and \( \tau_{S3} \) are the limiting values for, respectively, the advection-dominated, transient-dominated, and diffusion-dominated cases. We should also note that Eqs. (9)–(15) involve the ratios of matrix norms. Our experience has shown that these ratios are relatively insensitive to the definition of the norm.

Examples herein employ the Frobenius norm.

In [10], the element-vector-based \( \tau_{\text{SUPG}} \) is defined as

\[
\begin{align*}
\tau_{\text{SUPG}}(\nu) &= \left( \frac{1}{\tau_{S1}} + \frac{1}{\tau_{S2}} + \frac{1}{\tau_{S3}} \right)^{-1}, \quad (17)
\end{align*}
\]

where

\[
\begin{align*}
\tau_{SV1} &= \left\| \frac{c_v}{k_v} \right\|, \\
\tau_{SV2} &= \left\| \frac{c_v}{k_v} \right\|, \\
\tau_{SV3} &= \tau_{SV1} \text{Re} = \left( \frac{c_v}{k_v} \right) \text{Re}.
\end{align*}
\]

\[2.2 \quad \text{Navier-Stokes Equations of Incompressible Flows.} \]

The Navier-Stokes equations for incompressible flows can be written as

\[
\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u - f \right) - \nabla \cdot \sigma = 0 \quad \text{on } \Omega, \quad (21)
\]

\[
\nabla \cdot u = 0 \quad \text{on } \Omega, \quad (22)
\]

where \( \rho, u, \) and \( f \) are the density, velocity, and the external force, respectively. The stress tensor \( \sigma \) is defined as

\[
\sigma(p, u) = -p \mathbf{I} + 2 \mu \varepsilon(u). \quad (23)
\]

Here \( p \) is the pressure, \( \mathbf{I} \) is the identity tensor, \( \mu = \rho \nu \) is the viscosity, \( \nu \) is the kinematic viscosity, and \( \varepsilon(u) \) is the strain-rate tensor:

\[
\varepsilon(u) = \frac{1}{2} (\nabla u + (\nabla u)^T), \quad (24)
\]

The essential and natural boundary conditions associated with Eq. (21) are

\[
u = g \quad \text{on } \Gamma_g, \quad \mathbf{n} \cdot \sigma = h \quad \text{on } \Gamma_h, \quad (25)
\]

where \( g \) and \( h \) are given functions. A divergence-free velocity field \( u_0(x) \) is specified as the initial condition.

Given suitably defined finite-dimensional trial solution and test function spaces for velocity and pressure, \( S^h \), \( \mathbf{V}^h \), \( S^h \), and \( V^h \), the stabilized finite element formulation of Eqs. (21)–(22) can be written as follows: Find \( u^h \in S^h \) and \( p^h \in S^h \) such that \( \forall w^h \in \mathbf{V}^h \) and \( q^h \in V^h \):

\[
\int_{\Omega} w^h \cdot \rho \left( \frac{\partial u^h}{\partial t} + u^h \cdot \nabla u^h - f \right) d\Omega + \int_{\Omega} \varepsilon(w^h) : \sigma(p^h, u^h) d\Omega - \int_{\Gamma_h} w^h \cdot h^t d\Gamma
\]

\[
+ \int_{\Omega} q^h \nabla \cdot u^h d\Omega
\]

\[
+ \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\text{SUPG} \rho} u^h \cdot \nabla w^h + \tau_{\text{SUPG}} q^h d\Omega
\]

\[
+ \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\text{SUPG} \rho} u^h \cdot \nabla w^h + \tau_{\text{SUPG}} q^h d\Omega
\]

\[
+ \int_{\Omega} \tau_{\text{SUPG} \rho} u^h \cdot \nabla w^h + \tau_{\text{SUPG}} q^h d\Omega = 0. \quad (26)
\]

Here \( \tau_{\text{SUPG}} \) and \( \tau_{\text{LSIC}} \) are the PSPG and LSIC (least-squares on incompressibility constraint) stabilization parameters.
We now define the following element-level matrices and vectors:

\[ m = \int_{\Omega^e} w^h \cdot \frac{\partial u^h}{\partial t} \, d\Omega : m_v, \]  

\[ c = \int_{\Omega^e} w^h \cdot \rho (u^h \cdot \nabla u^h) \, d\Omega : c_v, \]  

\[ k = \int_{\Omega^e} (u^h \cdot \nabla w^h) \cdot \rho (u^h \cdot \nabla u^h) \, d\Omega : k_v, \]  

\[ g = \int_{\Omega^e} (\nabla \cdot w^h) p^h \, d\Omega : g_v, \]  

\[ g^T = \int_{\Omega^e} q^h (\nabla \cdot u^h) \, d\Omega : g_v^T, \]  

\[ k_\varepsilon = \int_{\Omega^e} (u^h \cdot \nabla w^h) \cdot \rho (u^h \cdot \nabla u^h) \, d\Omega : k_\varepsilon_v, \]  

\[ c_\varepsilon = \int_{\Omega^e} (u^h \cdot \nabla w^h) \cdot \rho \frac{\partial u^h}{\partial t} \, d\Omega : c_\varepsilon_v, \]  

\[ \gamma_v = \int_{\Omega^e} (u^h \cdot \nabla w^h) \cdot \nabla p^h \, d\Omega : \gamma_v, \]  

\[ \beta_v = \int_{\Omega^e} \nabla q^h \cdot \frac{\partial u^h}{\partial t} \, d\Omega : \beta_v, \]  

\[ \theta_v = \int_{\Omega^e} \nabla q^h \cdot \nabla p^h \, d\Omega : \theta_v, \]  

\[ e_v = \int_{\Omega^e} (\nabla \cdot w^h) p (\nabla \cdot u^h) \, d\Omega : e_v. \]

The element-level Reynolds and Courant numbers are defined in the same way as they were defined before, given by Eqs. (9)–(12). The components of the element-matrix-based \( \tau_{SUPG} \) are defined in the same way as they were defined before, given by Eqs. (13)–(15). \( \tau_{SUPG} \) is constructed from its components in the same way as it was constructed before, given by Eq. (16). The components of the element-vector-based \( \tau_{SUPG} \) are defined in the same way as they were defined before, given by Eqs. (18)–(20). The construction of \( (\tau_{SUPG})_v \) is also the same as it was before, given by Eq. (17).

From [10], we write the element-matrix-based \( \tau_{PSPG} \) as

\[ \tau_{PSPG} = \left( \frac{1}{\tau_{P1}} + \frac{1}{\tau_{P2}} + \frac{1}{\tau_{P3}} \right)^{-\frac{1}{2}}, \]  

where

\[ \tau_{P1} = \frac{\|\varepsilon\|}{\|\gamma\|}, \]  

\[ \tau_{P2} = \frac{\Delta t \|g\|}{\|\beta\|}, \]  

\[ \tau_{P3} = \tau_{P1} \text{Re} = \left( \frac{\|\varepsilon\|}{\|\gamma\|} \right) \text{Re}. \]

Also from [10], the element-vector-based \( \tau_{PSPG} \) is written as

\[ (\tau_{PSPG})_v = \left( \frac{1}{\tau_{P1}} + \frac{1}{\tau_{P2}} + \frac{1}{\tau_{P3}} \right)^{-\frac{1}{2}}, \]

where

\[ \tau_{PV1} = \tau_{P1}, \]  

\[ \tau_{PV2} = \frac{\|\varepsilon\|}{\|\gamma\|}, \]  

\[ \tau_{PV3} = \tau_{PV1} \text{Re}. \]

Lastly from [10], the element-matrix-based \( \tau_{LSIC} \) and the element-vector-based \( \tau_{LSIC} \) are given as

\[ \tau_{LSIC} = \frac{\|e\|}{\|\varepsilon\|}, \]  

\[ (\tau_{LSIC})_v = \tau_{LSIC}. \]

For the purpose of comparison, we also define here stabilization parameters that are based on an earlier definition of the length scale \( h \) first introduced in [7]:

\[ h_{UGN} = 2\|u^h\| \sum_{a=1}^{n_{fe}} \left| u^h \cdot \nabla N_a \right|^{-1}, \]

where \( N_a \) is the interpolation function associated with node \( a \). The stabilization parameters are defined as

\[ \tau_{SUGN1} = \frac{h_{UGN}}{2\|u^h\|}, \]  

\[ \tau_{SUGN2} = \frac{\Delta t \|g\|}{2\|\beta\|}, \]  

\[ \tau_{SUGN3} = \frac{h_{UGN}^2}{4\|u^h\|}, \]

\[ (\tau_{SUPG})_{UGN} = \left( \frac{1}{\tau_{SUGN1}} + \frac{1}{\tau_{SUGN2}} + \frac{1}{\tau_{SUGN3}} \right)^{-\frac{1}{2}}, \]

\[ (\tau_{PSPG})_{UGN} = (\tau_{SUPG})_{UGN}. \]

\[ (\tau_{LSIC})_{UGN} = \frac{h_{UGN}}{2\|u^h\|}. \]

Here \( \delta \) is given as follows:

\[ \delta = \left( \frac{\text{Re}_{UGN}}{\text{Re}_{UGN} \leq 3}, \frac{1}{\text{Re}_{UGN} > 3} \right), \]

where \( \text{Re}_{UGN} = \frac{\|u^h\|}{2\|u^h\|}. \)

Remark 1 The discontinuity-capturing directional dissipation (DCDD) stabilization was introduced in [11] as a potential alternative or complement to the LSIC stabilization. As part of the DCDD stabilization, a second element length scale that is based on the solution gradient was also introduced in [11].

Remark 2 New definitions for the diffusion-dominated limits of the SUPG and PSPG stabilization parameters were introduced in [12]. These new definitions are closely related to the second element length scale that was first introduced in [11] and later employed in [12] as a diffusion length scale.

Remark 3 For the advection-dominated limits of the SUPG and PSPG stabilization parameters, equivalent length scales can be defined by simply multiplying the stabilization parameter with \( 2\|u^h\| \).

For the comparative investigation we would like to carry out, we also provide here element length scales defined in other studies, based on the element shapes and advection field. For notational convenience, we first define the following unit vector:
The element length given in [6] for a quadrilateral element can be written as

$$h_{SA1} = \left( \frac{x_2 + x_3}{2} - \frac{x_1 + x_4}{2} \right) \cdot s + \left( \frac{x_1 + x_4}{2} - \frac{x_1 + x_2}{2} \right) \cdot s.$$

(58)

where $x_a$ is the nodal coordinate vector associated with node $a$.

For triangular elements, we use the following expression from [13]:

$$h_{SA1} = \frac{1}{4} \left[ |(x_2 - x_1) \cdot s | + |(x_3 - x_2) \cdot s | + |(x_1 - x_3) \cdot s | \right].$$

(59)

To write some of the other element lengths given in [13], we first define a special sign function:

$$SSgn(y) = \begin{cases} -1 & y \leq 0 \\ +1 & y > 0 \end{cases}.$$

(60)

and the streamwise components of the nodal “radial” position vectors:

$$\delta_a = (x_o - x_a) \cdot s,$$

(61)

where

$$x_a = \left( \sum_{a=1}^{n_{ux}} x_a \right) / n_{ux}. \tag{62}$$

The number of upstream and downstream element nodes can be expressed as

$$n_{uen} = \sum_{a=1}^{n_{ux}} \frac{1}{2} (1 - SSgn(\delta_a)),$$

(63)

$$n_{den} = \sum_{a=1}^{n_{ux}} \frac{1}{2} (1 + SSgn(\delta_a)). \tag{64}$$

Then one of the element lengths given in [13] can be written as

$$h_{SA1} = \left( \sum_{a=1}^{n_{ux}} \frac{1}{2} (1 + SSgn(\delta_a)) \delta_a \right) / n_{den},$$

$$- \left( \sum_{a=1}^{n_{ux}} \frac{1}{2} (1 - SSgn(\delta_a)) \delta_a \right) / n_{uen}. \tag{65}$$

Another one of the element lengths given in [13] can be written as

$$h_{SA1} = max(\delta_1, \delta_2, \ldots, \delta_{n_{ux}}) - min(\delta_1, \delta_2, \ldots, \delta_{n_{ux}}). \tag{66}$$

A third element length given in [13] is the node-based version of the one given by Eq. (65):

$$(h_{SA1})_a = \left| \delta_a - \left( \sum_{a=1}^{n_{ux}} \frac{1}{2} (1 - SSgn(\delta_a)) \delta_a \right) \right| / n_{uen}. \tag{67}$$

2.3 Streamline-Upwind/Petrov-Galerkin (SUPG) Stabilization and Smagorinsky Turbulence Viscosities. To compare the numerical viscosities generated by the SUPG stabilization with the eddy viscosity introduced by a Smagorinsky turbulence model, we first write the following “viscosity” based on the SUPG stabilization parameter:

$$\nu_{SUPG} = \tau_{SUPG} \| u \|^2.$$

(68)

The eddy viscosity introduced by a Smagorinsky turbulence model is based on the element length scales [15] is written as

$$\nu_{SMAG} = (0.1 h_{SMAG})^2 \left( 2 \varepsilon(u^h) : e(u^h) \right)^{1/2}. \tag{69}$$
and linear triangular elements, with $\|u\| = 1.0$ and $\Delta t = 1.0$, and as function of the advection direction. The test flow computations reported in [10] show that the definitions based on the element-level matrices and vectors perform well.

Here, element lengths are calculated and compared for linear, quadratic and cubic elements in two-dimensions, based on four of the definitions given in this paper: the equivalent length scale computed from $\tau_{S1}$ (with the Frobenius norm of the element level matrices), $h_{UGN}$, $h_{SA1}$, and $h_{SA2}$. Definitions that depend on the location within an element are evaluated at the origin of the natural coordinate system for quadrilateral elements and at the centroid for triangular elements. Both Lagrangian and serendipity quadrilaterals have been evaluated. The element lengths calculated based on the definitions listed above are shown in Figs. 5–7. The element shape is indicated by a dashed line and the nodes are indicated by a circled cross. Each closed curve represents a different element length definition. For each advection direction, the element length is that of a line through the element center, parallel to the advection, bounded by its intersections with the closed curve. In other words, let us imagine a line passing through the center and find its two intersection points with the closed curve. Then the distance between those two points is the element length in that advection direction. Although the results displayed here for $\tau_{S1}$ are based on the Frobenius norm of the element level matrices, we see little difference between the $\tau_{S1}$ calculated with different matrix norms. From Figs. 5–7, we note that the difference between different element length definitions is more pronounced for higher-order elements. In general, the element length decreases with the increase in the order of the element. This observation is consistent with what we see for $h_{UGN}$ in one dimension (see Fig. 4).

3.2 Comparison of Streamline-Upwind/Petrov-Galerkin (SUPG) Stabilization and Smagorinsky Turbulence Viscosities. Flow past a cylinder at $Re = 3,000$ and $Re = 50,000$ are used as test problems to compare the numerical viscosities generated by the SUPG stabilization with the numerical viscosity introduced by a Smagorinsky turbulence model. The stabilization parameters...
are computed as given by Eqs. (49)–(56), but with the \( r_{\text{SUPG}} \) component dropped. When calculating \( \text{Re}_{\text{UGN}} \) used in Eq. (56), the kinematic viscosity \( \nu \) is augmented with \( \nu_{\text{SMAG}} \). Velocity and pressure are both interpolated with bilinear functions. A mesh with 14,960 nodes and 14,700 quadrilateral elements is employed. Close to the cylinder surface, the radial distance between the mesh points (normalized by the cylinder diameter) is \( 2.5 \times 10^{-2} \) at \( \text{Re} = 3,000 \) and \( 5 \times 10^{-3} \) at \( \text{Re} = 50,000 \). A close-up view of the mesh for the latter case is shown in Fig. 8. In each case, the computations are carried out until a developed unsteady solution is obtained. Then, based on the velocity field at a given instant, \( \nu_{\text{SMAG}} / \nu_{\text{SUPG}} \) is calculated. Figure 9 shows the vorticity and \( \nu_{\text{SMAG}} / \nu_{\text{SUPG}} \) for \( \text{Re} = 3,000 \). Shades of gray represent values of \( \nu_{\text{SMAG}} / \nu_{\text{SUPG}} \) ranging from 0.00 (white) to 0.05, with black indicating 0.05 and higher values. Except for the regions in black, \( \nu_{\text{SMAG}} / \nu_{\text{SUPG}} \) is less than 5%. Because Fig. 9 shows pictures zoomed into a small part of the full domain, one can also infer that most of the full domain is marked in white, and therefore for those regions the ratio is essentially 0%. As additional information, we would like to note that when we inspect the overall data for \( \nu_{\text{SMAG}} / \nu_{\text{SUPG}} \), we see that in most of the domain it is less than 1%. The turbulence model is active only in regions with significant vorticity. Except for a very few points in the near wake, \( \nu_{\text{SUPG}} \) dominates \( \nu_{\text{SMAG}} \). When a wall damping function is used with the turbulence model, \( \nu_{\text{SMAG}} \) becomes even smaller. Similar observations can be made for \( \text{Re} = 50,000 \) (see Fig. 10). It is important to remember that while \( \nu_{\text{SMAG}} \) is an isotropic viscosity, \( \nu_{\text{SUPG}} \) is the maximum value of a directional viscosity, with the maximum value attained in the advection direction. However, in most of the domain \( \nu_{\text{SMAG}} / \nu_{\text{SUPG}} \) is so small that, except for directions nearly perpendicular to the advection direction, \( \nu_{\text{SMAG}} \) will still be substantially less than the direction-adjusted value of \( \nu_{\text{SUPG}} \). It is also important to remember that \( \nu_{\text{SUPG}} \) is generated by a residual-based formulation, while \( \nu_{\text{SMAG}} \) is not.

---

**Fig. 6** For a square serendipity element, the element length calculated with different definitions and as function of advection direction

**Fig. 7** For an equilateral triangular element, the element length calculated with different definitions and as function of advection direction

**Fig. 8** Flow past a cylinder. A close-up view of the finite element mesh with 14,960 nodes and 14,700 elements.
4 Concluding Remarks

For the SUPG and PSPG formulations for flow problems, we presented a comparative study of the element lengths — i.e., local length scales — defined in different ways. These element lengths are closely related to the stabilization parameters used in the SUPG and PSPG formulations. Our comparisons included parameters defined based on the element-level matrices and vectors, some earlier definitions of element lengths, and extensions of these to higher-order elements. This comparative study shows that the difference between different element length definitions is more pronounced for higher-order elements, and the element length decreases with the increase in the order of the element. We believe that the parameter definitions based on the element-level matrices and vectors provide a good, general framework that automatically takes into account the local length scales and the advection field. Therefore these stabilization parameter definitions and the corresponding element length definitions are what we favor. We also compared, based on some test flow computations, the numerical viscosities generated by the SUPG stabilization with the eddy viscosity associated with a Smagorinsky turbulence model. These test computations show that, in most of the flow domain, the SUPG viscosity is much larger than the Smagorinsky viscosity. It is clear that better understanding is needed for the performance of the stabilized formulations with higher-order element and also for the interaction between the stabilized formulations and the Smagorinsky turbulence model.

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