A proper finite element formulation is derived for the vorticity-stream function form of the incompressible Euler equations on two-dimensional multiply-connected domains. The formulation includes the variational equation needed for determining the value of the stream function at the internal boundaries.

1. Introduction

In the vorticity-stream function formulation of two-dimensional incompressible flow problems numerical treatment of the boundary conditions associated with solid surfaces calls for particular care. The situation becomes especially interesting in the case of multiply-connected domains.

For problems governed by the incompressible Navier-Stokes equations, at no-slip boundaries corresponding to solid surfaces there are two boundary conditions for the stream function but none for the vorticity. The variational equation needed to compensate for the lack of boundary condition for the vorticity at such boundaries can be derived from the Poisson equation for the stream function [1]. This issue was also addressed in [2-5]. In the case of multiply-connected domains the value of the stream function at the internal boundaries is also unknown. The additional variational equations needed can be derived by integrating the equation of motion in the velocity-pressure form along each internal boundary and combining it with the vorticity transport equation [1].

In this paper, proper finite element equations are derived for problems governed by the incompressible Euler equations. The approach is similar to the one taken for the Navier-Stokes equations; but this time one does not need to be concerned about any lack of boundary conditions for the vorticity at solid surfaces. Nevertheless, in the case of multiply-connected domains the value of the stream function at the internal boundaries is unknown and must be determined as part of the overall solution.

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Let us assume that the normal component of the velocity is zero and, therefore, the value of the stream function is constant along each internal boundary but still a function of time for time-dependent problems. By integrating the equation of motion in the velocity-pressure form, it can be shown that the integral of the tangential component of the velocity along an internal boundary does not change with time. This result can be incorporated into the variational formulation of the Poisson equation for the stream function; the outcome is the variational equation needed for determining the value of the stream function at an internal boundary. The details of the procedure are described in the following sections.

2. Problem statement

Let Ω be a domain in \( \mathbb{R}^2 \) and \( T \) be a positive real number. The spatial and temporal coordinates will be denoted by \( x \in \Omega \) and \( t \in [0, T] \) where a superposed bar indicates the set closure. The boundary \( \Gamma \) of the domain \( \Omega \) consists of an external boundary \( \Gamma_0 \) and, possibly, several internal boundaries denoted by \( \Gamma_k, k = 1, 2, \ldots, q \), i.e.,

\[
\Gamma = \bigcup_{k=0}^{q} \Gamma_k ,
\]

where \( q \) is the number of internal boundaries.

The Euler equations for incompressible flow can be written as

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + \frac{1}{\rho} \nabla p = 0 \quad \text{on } \Omega \times (0, T) , \tag{2.2}
\]

\[
\nabla \cdot u = 0 \quad \text{on } \Omega \times (0, T) , \tag{2.3}
\]

where \( u, p, \) and \( \rho \) are the velocity, pressure, and density, respectively. The incompressibility constraint of equation (2.3) is satisfied automatically by the definition of the stream function \( \psi \):

\[
u_1 = \frac{\partial \psi}{\partial x_2} , \quad u_2 = -\frac{\partial \psi}{\partial x_1} . \tag{2.4a, b}
\]

The momentum equation given by (2.2) can be translated into a convective transport equation for the (scalar) vorticity

\[
\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} , \tag{2.5}
\]

by taking the curl of (2.2), that is,

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = 0 \quad \text{on } \Omega \times (0, T) . \tag{2.6}
\]

A Poisson equation involving the vorticity and the stream function can be obtained by eliminating the velocity between equations (2.4) and (2.5):

\[
\nabla^2 \psi + \omega = 0 \quad \text{on } \Omega \times (0, T) . \tag{2.7}
\]
REMARK 2.1. In the context of inviscid fluid flow the internal boundaries \((\Gamma_k, k = 1, 2, \ldots, q)\) represent the solid obstacles with no surface friction.

REMARK 2.2. On the boundaries the tangential and normal velocity components \(u_t\) and \(u_n\) can be expressed as the normal and tangential derivatives of the stream function, that is:

\[
\begin{align*}
  u_t &= \tau \cdot u = n \cdot \nabla \psi = \frac{\partial \psi}{\partial n}, \\
  u_n &= n \cdot u = -\tau \cdot \nabla \psi = -\frac{\partial \psi}{\partial \tau},
\end{align*}
\]

where \(\tau\) and \(n\) are the unit tangential and normal vectors (see Fig. 1).

**Boundary conditions for the external boundary**

We assume that the external boundary \(\Gamma_0\) admits the following decomposition with respect to the type of boundary conditions specified for \(\psi\).

\[
\Gamma_0 = \Gamma_g \cup \Gamma_G \cup \Gamma_n,
\]

\[
\emptyset = \Gamma_g \cap \Gamma_G, \quad \emptyset = \Gamma_G \cap \Gamma_n, \quad \emptyset = \Gamma_n \cap \Gamma_g.
\]

Here \(\Gamma_g\) and \(\Gamma_n\) denote the subsets of \(\Gamma_0\) where the specified condition for \(\psi\) is Dirichlet and Neumann type, respectively. On \(\Gamma_G\) both Dirichlet and Neumann type conditions are specified for \(\psi\).

REMARK 2.3. Specifying \(\psi\) is equivalent to specifying the normal component of the velocity. This can be done, for example, if the boundary is a known streamline. We assume that for every external solid boundary the associated streamline is known.

REMARK 2.4. On \(\Gamma_G\) both components of the velocity are specified. At an external boundary at least one component of the velocity must be specified. The second component can be specified as a replacement for specifying \(\omega\), but only if we are allowed to specify \(\omega\) in the first place. Because the equation governing the transport of \(\omega\) is hyperbolic, \(\omega\) is allowed to be specified at a boundary only if the flow at that boundary is inward, that is,

\[
x \in \Gamma_G \Rightarrow u_n(x) < 0.
\]

![Fig. 1. Unit normal and tangential vectors.](image-url)
A similar decomposition of \( \Gamma_0 \) can be made with respect to \( \omega \).

\[
\Gamma_0 = \Gamma_G \cup \Gamma_G \cup \Gamma_h ,
\]

\[
\emptyset = \Gamma_G \cap \Gamma_G , \quad \emptyset = \Gamma_G \cap \Gamma_h , \quad \emptyset = \Gamma_h \cap \Gamma_h .
\]

Here \( \Gamma_\emptyset \) denotes the part of the boundary where we are allowed to specify \( \omega \) and we do so. We have already stated that \( \omega \) can be specified at a boundary only if the flow at that boundary is inward (see Remark 2.4). Therefore,

\[
x \in \Gamma_\emptyset \Rightarrow u_n(x) < 0 .
\]

Furthermore we can state that if at a boundary the flow is inward that boundary must belong to either \( \Gamma_\emptyset \) or \( \Gamma_G \), that is,

\[
u_n(x) < 0 \Rightarrow x \in \Gamma_G \cup \Gamma_\emptyset .
\]

We can say this in another way: the part of the boundary at which the flow is not inward (this includes the cases of the flow being outward or the boundary being a streamline) must belong to \( \Gamma_h \). Note that on \( \Gamma_G \) we are allowed to specify \( \omega \) but we choose not to do so (see Remark 2.4).

The following set of boundary conditions are given at the external boundary:

\[
\psi(x, t) = g(x, t) \quad \text{on } \Gamma_\emptyset \times (0, T) ,
\]

\[
\psi(x, t) = G(x, t) \quad \text{on } \Gamma_G \times (0, T) ,
\]

\[
\frac{\partial \psi}{\partial n}(x, t) = H(x, t) \quad \text{on } \Gamma_G \times (0, T) ,
\]

\[
\frac{\partial \psi}{\partial n}(x, t) = h(x, t) \quad \text{on } \Gamma_h \times (0, T) ,
\]

\[
\omega(x, t) = \tilde{g}(x, t) \quad \text{on } \Gamma_\emptyset \times (0, T) ,
\]

where the functions \( g(x, t) \), \( G(x, t) \), \( H(x, t) \), \( h(x, t) \), and \( \tilde{g}(x, t) \) are assumed to be known.

**Boundary conditions for the internal boundaries**

Let us assume that at all internal boundaries the normal component of the velocity vanishes, that is,

\[
-\frac{\partial \psi}{\partial \tau}(x, t) = 0 \quad \text{on } \Gamma_k \times (0, T) , \quad k = 1, 2, \ldots , q .
\]

The value of the stream function is constant along each internal boundary but is still unknown (possibly a function of time) and must be determined as part of the overall solution. To
determine such unknown values of the stream function we consider (2.2) along each internal boundary:

$$\frac{\partial u_{\tau}}{\partial t} + u_{\tau} \frac{\partial u_{\tau}}{\partial \tau} + \frac{1}{\rho} \frac{\partial p}{\partial \tau} = 0 \quad \text{on } \Gamma_k \times (0, T), \ k = 1, 2, \ldots, q. \quad (2.21)$$

Integrating (2.21) along $$\Gamma_k$$ we obtain

$$\frac{\partial}{\partial t} \int_{\Gamma_k} u_{\tau} \, d\Gamma + \int_{\Gamma_k} \frac{\partial}{\partial \tau} \left( \frac{1}{2} u_{\tau}^2 + \frac{p}{\rho} \right) \, d\Gamma = 0 \quad \text{on } (0, T), \ k = 1, 2, \ldots, q. \quad (2.22)$$

Assuming that $$\left( \frac{1}{2} u_{\tau}^2 + \frac{p}{\rho} \right)$$ is single valued, the second integral on the left-hand side of equation (2.22) vanishes identically; consequently we get

$$\int_{\Gamma_k} u_{\tau} \, d\Gamma = (\text{constant})_k \quad \text{on } (0, T), \ k = 1, 2, \ldots, q. \quad (2.23)$$

This is the additional equation needed to determine the unknown value of the stream function at an internal boundary. For time-dependent problems the constants in (2.23) are evaluated from the given initial condition; for example, if the flow field is initially stagnant than all such constants have zero values. For steady-state problems the constants in (2.23) must be given.

The initial condition for this problem consists of specification of the initial distribution of the vorticity, that is,

$$\omega(x, 0) = \omega_0(x) \quad \text{on } \Omega, \quad (2.24)$$

where $$\omega_0(x)$$ is a given function.

3. The finite element formulation

Let $$(\mathcal{E})$$ denote the set of all elements emanating from the finite element discretization of the computational domain $$\Omega$$ into subdomains $$\Omega^e, \ e = 1, 2, \ldots, n_{el},$$ such that

$$\bar{\Omega} = \bigcup_{e=1}^{n_{el}} \bar{\Omega}^e, \quad (3.1)$$

$$\emptyset = \bigcap_{e=1}^{n_{el}} \Omega^e, \quad (3.2)$$

where $$n_{el}$$ is the number of elements. Let $$\Gamma^e$$ denote the boundary of $$\Omega^e$$. We associate to $$(\mathcal{E})$$ the following finite dimensional spaces:

$$H^{1h} = \{ \phi^h \mid \phi^h \in C^0(\bar{\Omega}), \phi^h \mid_{\partial \Omega} \in P^1 \ \forall \Omega^e \in \mathcal{E} \}, \quad (3.3)$$

$$S^h = \left\{ \psi^h \mid \psi^h \in H^{1h}, \psi^h = g \ \text{on } \Gamma_k, \psi^h = G \ \text{on } \Gamma_G, \frac{\partial \psi^h}{\partial \tau} = 0 \ \text{on } \Gamma_k, k = 1, 2, \ldots, q \right\}, \quad (3.4)$$
The finite element formulation associated with equation (2.6) is given as follows: Find \( \psi^h \in S^h \) and \( \omega^h \in \tilde{S}^h \), such that

\[
\int_{\Omega} \left( w^h + \delta^h \right) \left( \frac{\partial \omega^h}{\partial t} + u \cdot \nabla \omega \right) \, d\Omega = 0 \quad \forall w^h \in \tilde{V}^h,
\]

where \( \delta^h \) is a \( C^{-1}(\Omega) \) Petrov–Galerkin supplement to the weighting function \( w^h \).

**REMARK 3.1.** The term \( \delta^h \) needs to act only in the element interiors and therefore is allowed to be discontinuous across element boundaries. Setting this term to zero reduces the formulation to a (Bubnov–)Galerkin one.

**REMARK 3.2.** Various Petrov–Galerkin procedures, particularly those based on the stream-line-upwind/Petrov–Galerkin (SUPG) formulations, have been successfully applied to a wide range of fluid flow problems. For an extensive review and discussion on the subject see [6].

For completeness we briefly describe the SUPG formulation employed here. In this formulation the SUPG supplement term, which can depend upon temporal as well as spatial discretization, is defined as follows [7]:

\[
\delta^h = C_2 \frac{h}{2} s \cdot \nabla w^h,
\]

with

\[
s = \frac{u}{\|u\|},
\]

and

\[
h = 2 \left( \sum_{a=1}^{n_{en}} |s \cdot \nabla N_a| \right)^{-1} \quad \text{('element length')},
\]
where \( N_a \) is the bilinear function (associated with the element node \( a \)) leading to a Galerkin formulation and \( n_{en} \) is the number of element nodes. The algorithmic Courant number \( C_{2r} \) (see [7]), can be selected as

\[
C_{2r} = 1 \quad \text{or} \quad C_{2r} = \min(C_{\Delta t}, 1), \tag{3.14a, b}
\]

where \( C_{\Delta t} = \Delta t \| u \| / h \) is the element Courant number. The second selection results in dependence upon temporal as well as spatial discretization.

**REMARK 3.3.** It can be shown that for elements with Courant number less than one the expression (3.14b) leads to symmetric positive-definite systems [8]. If we replace this expression with \( C_{2r} = C_{\Delta t} \), then we obtain symmetric positive-definite systems for all the elements at any Courant number.

The discrete variational formulation associated with (2.7) can be stated as follows: Find \( \psi^h \in S^h \) and \( \omega^h \in \tilde{S}^h \), such that

\[
\int_{\Omega} \nabla w^h \cdot \nabla \psi^h \, d\Omega - \int_{\Omega} w^h \omega^h \, d\Omega = \int_{\Gamma_k} w^h \theta^h \, d\Gamma \quad \forall \psi^h \in V^h \tag{3.15}
\]

and

\[
\int_{\Omega} \nabla w^h \cdot \nabla \psi^h \, d\Omega - \int_{\Omega} w^h \omega^h \, d\Omega = \int_{\Gamma_k} w^h \theta^h \, d\Gamma + \int_{\Gamma_G} w^h H \, d\Gamma \quad \forall \psi^h \in V^h \tag{3.16}
\]

Corresponding to (2.23) we write the following discrete variational formulation: Find \( \psi^h \in S^h \) and \( \omega^h \in \tilde{S}^h \), such that

\[
\int_{\Omega} \nabla w^h \cdot \nabla \psi^h \, d\Omega - \int_{\Omega} w^h \omega^h \, d\Omega = \int_{\Gamma_k} w^h \theta^h \, d\Gamma \quad \forall \psi^h \in V^h_{kR}, \quad k = 1, 2, \ldots, q. \tag{3.17}
\]

Since \( w^h \) is constant along \( \Gamma_k \), we can rewrite (3.17) as

\[
\int_{\Omega} \nabla w^h \cdot \nabla \psi^h \, d\Omega - \int_{\Omega} w^h \omega^h \, d\Omega = w^h(\Gamma_k) \int_{\Gamma_k} \theta^h \, d\Gamma. \tag{3.18}
\]

Combining (3.18) with (2.23) we get

\[
\int_{\Omega} \nabla w^h \cdot \nabla \psi^h \, d\Omega - \int_{\Omega} w^h \omega^h \, d\Omega = w^h(\Gamma_k)(\text{constant})_k, \tag{3.19}
\]

where the constants are evaluated as described in Section 2. The following equation can be written as a special case of (3.19):

\[
w^h(\Gamma_k)(\text{constant})_k = \int_{\Omega} \nabla w^h \cdot \nabla \psi^h_0 \, d\Omega - \int_{\Omega} w^h \omega^h_0 \, d\Omega; \tag{3.20}
\]

here \( \psi^h_0 \) and \( \omega^h_0 \) denote the initial values of the stream function and the vorticity.
The functions $\psi^h$ and $\omega^h$ can be written as sums of their component functions:

$$\psi^h = \psi^h_\varepsilon + \psi^h_\varepsilon + \sum_{k=1}^{q} \psi^h_k,$$

$$\omega^h = \omega^h_\varepsilon + \omega^h_\varepsilon + \omega^h_\varepsilon,$$

in which the unknowns are $\psi^h_\varepsilon$, $\psi^h_k$ ($k = 1, 2, \ldots, q$), $\omega^h_\varepsilon$, and $\omega^h_\varepsilon$. Note that

$$\psi^h_\varepsilon \equiv g \quad \text{on } \Gamma^h_\varepsilon,$$  

$$\psi^h_\varepsilon \equiv G \quad \text{on } \Gamma^h_\varepsilon,$$  

$$\omega^h_\varepsilon \equiv \tilde{g} \quad \text{on } \Gamma^h_\varepsilon,$$

and

$$\psi^h \in V^h_*,$$  

$$\psi^h_k \in V^h_{kR}, \quad k = 1, 2, \ldots, q,$$  

$$\omega^h_\varepsilon \in V^h_*,$$  

$$\omega^h_\varepsilon \in \tilde{V}^h_*.$$

The matrix formulation associated with (3.10) can be written as

$$\tilde{M}(d_*, d_q)d_\ast + \tilde{C}(d_*, d_q)v_\ast + \tilde{B}(d_*, d_q, v_G, a_G) = \tilde{F},$$

while the matrix formulation corresponding to (3.15), (3.16) and (3.19) is

$$K_*d_\ast + K_qd_q - M_*v_\ast - M_Gv_G = F,$$

where $v_\ast$, $a_\ast$, $v_G$, $a_G$, $d_\ast$, and $d_q$ are the vectors of the nodal values of $w^h_\ast$, $\partial w^h_\ast/\partial t$, $w^h_\ast$, $\partial w^h_G/\partial t$, $\psi^h_\ast$ and $\psi^h_k$ ($k = 1, 2, \ldots, q$), respectively. The vectors $\tilde{B}$, $\tilde{F}$, and $F$ as well as the matrices $\tilde{M}$, $\tilde{C}$, $K_*$, $K_q$, $M_*$, and $M_G$ are derived from (3.10), (3.15), (3.16) and (3.19).

The equation system given by (3.31) can also be written as three separate equation systems corresponding to (3.15), (3.16) and (3.19), respectively:

$$(K_*)_{\text{II}}d_\ast + (K_q)_{\text{II}}d_q - (M_*)_{\text{II}}v_\ast - (M_G)_{\text{II}}v_G = F_{\text{II}},$$

$$(K_*)_{\text{III}}d_\ast + (K_q)_{\text{III}}d_q - (M_*)_{\text{III}}v_\ast - (M_G)_{\text{III}}v_G = F_{\text{III}},$$

$$(K_*)_{\text{IV}}d_\ast + (K_q)_{\text{IV}}d_q - (M_*)_{\text{IV}}v_\ast - (M_G)_{\text{IV}}v_G = F_{\text{IV}},$$

where the subscripts II, III and IV denote the submatrices associated with (3.15), (3.16) and (3.19), respectively. A generalized trapezoidal algorithm can be used for time integration:
\[(\mathbf{v}_*)_{n+1} = (\mathbf{v}_*)_n + (1 - \alpha) \Delta t (\mathbf{a}_*)_n + \alpha \Delta t (\mathbf{a}^\ast)_n, \quad (3.35)\]
\[(\mathbf{v}_G)_{n+1} = (\mathbf{v}_G)_n + (1 - \alpha) \Delta t (\mathbf{a}_G)_n + \alpha \Delta t (\mathbf{a}^\ast)_n, \quad (3.36)\]

where the parameter \(\alpha\) controls the stability and accuracy of the algorithm.

There are various ways \([1, 8]\) to solve (3.30–3.36); some are more iterative in nature than others. For example, in a typical block-iteration scheme \(\mathbf{v}_*, \mathbf{d}_*, \mathbf{v}_G, \text{ and } d_q\) can be treated as unknown in (3.30), (3.32), (3.33) and (3.34), respectively.

4. Conclusion

In this paper the derivation of appropriate finite element equations has been presented for the two-dimensional problems governed by the vorticity-stream function form of the incompressible Euler equations. In the case of multiply-connected domains the value of the stream function at an internal boundary is determined by integrating the equation of motion in the velocity-pressure form along that boundary and then incorporating the result into the variational formulation of the Poisson equation for the stream function.

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References