COMPUTATION OF SPATIALLY PERIODIC FLOWS BASED ON THE VORTICITY–STREAM FUNCTION FORMULATION*

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Finite element solution strategies are presented for the two-dimensional, spatially periodic, viscous and inviscid, incompressible flows governed by the vorticity–stream function formulation. These strategies are successfully tested on various uniperiodic and biperiodic viscous flow problems involving arrays of cylinders with Reynolds number 0 and 100. It is shown that in all cases for Reynolds number 100 the solution becomes unsteady and ceases to satisfy the symmetry conditions along the horizontal centerlines of the cylinders.

1. Introduction

In this paper we describe our numerical solution procedures for the two-dimensional, spatially periodic, unsteady flows governed by the vorticity–stream function formulation of the incompressible Navier–Stokes and Euler equations. These procedures are based on the finite element method and therefore are conveniently applicable to problems involving intricate geometries.

We consider not only flows which are periodic in one direction (i.e. uniperiodic), but also flows which are periodic in both directions (i.e. biperiodic). It is assumed that the problem domains considered have rectangular external boundaries; however, any number of internal boundaries (i.e. obstacles) with arbitrary shapes may be involved. The way these internal boundaries are handled numerically is somewhat independent of whether the flow has spatial periodicity or not. In either case, for each internal boundary an additional equation based on the continuity of pressure is used to derive the variational formulation needed to solve for the unknown value of the stream function at that boundary [1, 2]. Therefore, contrary to the assessment made by Cerutti et al. [3], we are convinced that the vorticity–stream function formulation does not have any limitation for this type of problems.

We found in recent literature several examples of computations involving arrays of cylinders with the assumption that symmetry conditions can be imposed along the horizontal (i.e., parallel to the main flow direction) centerlines of the cylinders [4–6]. It is quite obvious to us that we should be as cautious with such assumptions as we are with the assumption of

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symmetry conditions along the horizontal centerline in the classical problem of flow about a single cylinder (i.e., the problem in which vortex shedding occurs for Reynolds numbers beyond \(-40\)). Therefore our computations are based on the true implementation of the spatial periodicity.

For uniperiodic flows we can show that the pressure difference between the upper and lower boundaries is invariant along the horizontal direction; of course for biperiodic flows similar statements can be made for both directions. Therefore, in addition to specifying the total horizontal (i.e. main) flow rate we can specify either the total vertical flow rate or the pressure difference between the upper and lower boundaries. We also show that for biperiodic flows we must discard one of the additional equations derived to solve for the unknown values of the stream function at the internal boundaries. The selection of the discarded equation is arbitrary. In our implementation, whichever equation is discarded, we set the value of the stream function along the corresponding internal boundary equal to zero.

Uniperiodic and biperiodic flow problems involving arrays of cylinders are considered in this paper as numerical examples. The purpose is not to make an in-depth physical study, but to test the formulation, show the existence of unsteady, nonsymmetric flow patterns and thus make our point about how this type of problems should be solved, and lay the groundwork for our future physical studies.

2. The vorticity–stream function formulation of spatially periodic flows

Let us consider a two-dimensional spatial domain \( \Omega \) which is, in general, multiply-connected and a time interval \((0, T)\) with \( x \) and \( t \) representing the coordinates associated with \( \Omega \) and \((0, T)\). The boundary \( \Gamma \) of the domain \( \Omega \) consists of an external boundary \( \Gamma_0 \) and \( q \) internal boundaries denoted by \( \Gamma_k, \quad k = 1, 2, \ldots, q \). We assume that the external boundary \( \Gamma_0 \) is of rectangular shape and has four subsets: \( \Gamma_{01}, \Gamma_{02}, \Gamma_{03} \) and \( \Gamma_{04} \). The upstream and downstream boundaries are represented by \( \Gamma_{01} \) and \( \Gamma_{03} \), while the upper and lower boundaries are represented by \( \Gamma_{02} \) and \( \Gamma_{04} \) (see Fig. 1). The unit normal and tangential vectors at a boundary are denoted by \( n \) and \( \tau \).

The vorticity–stream function formulation of the incompressible Navier–Stokes equations consists of a convection–diffusion equation for the vorticity and a Poisson equation for the stream function:

\[
\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega - \nu \nabla^2 \omega = 0 \quad \text{on} \quad \Omega \times (0, T),
\]

\[
\nabla^2 \psi + \omega = 0 \quad \text{on} \quad \Omega \times (0, T),
\]

with the stream function and vorticity defined as

\[
\mathbf{u} = \{ \partial \psi / \partial x_2, -\partial \psi / \partial x_1 \},
\]

\[
\omega = \partial u_2 / \partial x_1 - \partial u_1 / \partial x_2,
\]

where \( \mathbf{u} \) is the velocity and \( \nu \) is the kinematic viscosity. The components of the velocity vector...
$u$ in the directions denoted by the unit vectors $n$ and $\tau$ can be expressed as

$$u_\| = n \cdot u = -\partial \psi / \partial \tau ,$$

$$u_{\tau} = \tau \cdot u = \partial \psi / \partial n .$$

At the internal boundaries the normal component of the velocity is specified:

$$\partial \psi / \partial \tau = 0 \quad \text{on } \Gamma_k \times (0, T), \quad k = 1, 2, \ldots, q ;$$

for viscous flows the tangential component is also specified:

$$\partial \psi / \partial n = (u_{\tau})_k \quad \text{on } \Gamma_k \times (0, T), \quad k = 1, 2, \ldots, q ,$$

where $(u_{\tau})_k$ is a given function on the $k$th segment of the boundary $\Gamma$.

From (7) it is known that the value of the stream function is invariant along an internal boundary; however, it is not known what that value is and therefore an additional equation is needed for each internal boundary to determine the unknown value of the stream function. These additional equations are obtained by imposing the condition that pressure must be continuous along all closed contours. For viscous flows, assuming that $(u_{\tau})_k$ is invariant along $\Gamma_k$, it can be shown that [1]

$$\nu \int_{\Gamma_k} \partial \omega / \partial n \, d\Gamma = -S_k \partial (u_{\tau})_k / \partial t \quad \text{on } (0, T), \quad k = 1, 2, \ldots, q ,$$

where $S_k$ is the length of the boundary $\Gamma_k$. For inviscid flows we get [2]
\[ \int_{\Gamma_k} u_r \, d\Gamma = (\text{constant})_k \quad \text{on } (0, T), \quad k = 1, 2, \ldots, q, \quad (10) \]

where the constants in (10) are given.

We consider two types of periodic flows: uniperiodic and biperiodic. Uniperiodic flows are defined to be periodic only in one direction, e.g. in the \( x_2 \) direction, whereas biperiodic flows are defined to be periodic in both \( (x_1 \text{ and } x_2) \) directions.

**Unperiodic flows**

To impose the periodicity in the \( x_2 \) direction the following constraints are implemented:

\[ \omega(x_1, (x_2)_I, t) = \omega(x_1, (x_2)_I, t) \quad \forall x_1 \in ((x_1)_I, (x_1)_I), \quad \forall t \in (0, T), \quad (11) \]

\[ \psi(x_1, (x_2)_I, t) = \psi(x_1, (x_2)_I, t) + \Delta \psi(t) \quad \forall x_1 \in ((x_1)_I, (x_1)_I), \quad \forall t \in (0, T), \quad (12) \]

where \( \Delta \psi(t) \) is the total flow rate specified between points II and I.

At the upstream boundary we specify both components of the velocity, i.e.,

\[ \frac{\partial \psi}{\partial x_2} = U_1(t) \quad \text{on } \Gamma_{01} \times (0, T), \quad (13) \]

\[ -\frac{\partial \psi}{\partial x_1} = U_2(t) \quad \text{on } \Gamma_{01} \times (0, T), \quad (14) \]

where \( U_1 \) and \( U_2 \) are given functions of time. Usually \( U_2 \) is zero and \( U_1 \) is uniform on \( \Gamma_{01} \).

At the downstream boundary we impose the same condition as the one given by (13); in addition to this condition we can set the normal derivative of either the vorticity or the stream function to zero. Furthermore we specify the value of the stream function at point IV relative to point I, i.e.,

\[ \psi_{IV} = \psi_I + \delta \psi(t), \quad (15) \]

where \( \delta \psi(t) \) is the total (downward) flow rate between points I and IV; we set \( \delta \psi(t) \) to zero.

The values of the pressure at the upper and lower computational boundaries are related as described by the following equation:

\[ p(x_1, (x_2)_I, t) = p(x_1, (x_2)_I, t) + \delta p(t) \quad \forall x_1 \in ((x_1)_I, (x_1)_I), \quad \forall t \in (0, T), \quad (16) \]

where \( \delta p(t) \) is the pressure difference between points II and I. Note that (16) and (12) are very similar.

To prove (16) we consider a closed contour \( A-B-C-D-A \) (see Fig. 1). Because the continuity of the pressure along all closed contours is assured by either (9) or (10), we can write

\[ (p_B - p_A) - (p_C - p_D) = - \int_B^C \frac{\partial p}{\partial \tau} \, d\tau - \int_D^A \frac{\partial p}{\partial \tau} \, d\tau. \quad (17) \]
By noting that $dr = dx_1$ along $B-C$ and $dr = -dx_1$ along $D-A$ we rewrite (17) as

$$(p_B - p_A) - (p_C - p_D) = -\int_B^C \frac{\partial p}{\partial x_1} \, dx_1 + \int_A^D \frac{\partial p}{\partial x_1} \, dx_1.$$  

(18)

From the momentum equation in $x_1$ direction an expression for $\partial p/\partial x_1$ along $B-C$ and $A-D$ can be given in terms of the vorticity and the stream function:

$$-\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial x_1} = \partial(\partial \psi/\partial x_2)/\partial t + \frac{1}{2}\partial((\partial \psi/\partial x_2)^2 + (\partial \psi/\partial x_1)^2)/\partial x_1$$

$$+ \partial \psi/\partial x_1 \omega + \nu \partial \omega/\partial x_2.$$  

(19)

It is clear from this expression that for uniperiodic flows the values of $\partial p/\partial x_1$ at the corresponding points along $B-C$ and $A-D$ are equal. Therefore the right-hand side of (18) vanishes identically and we get

$$(p_C - p_D) = (p_B - p_A).$$  

(20)

Of course this is just another way of expressing (16).

Because of (16), as an alternative to specifying $\delta \psi(t)$ (to be zero), we can specify $\delta p(t)$ (to be zero). In other words we can set (equal to zero) either the total downward flow rate between points I and IV or the pressure difference between the upper and lower computational boundaries.

**Biperiodic flows**

We impose the periodicity in the $x_1$ direction the same way as we do in the $x_2$ direction:

$$\omega((x_1)_{IV}, x_2, t) = \omega((x_1)_I, x_2, t) \quad \forall x_2 \in ((x_2)_I, (x_2)_{II}) \quad \forall t \in (0, T),$$

(21)

$$\psi((x_1)_{IV}, x_2, t) = \psi((x_1)_I, x_2, t) + \delta \psi(t) \quad \forall x_2 \in ((x_2)_I, (x_2)_{II}) \quad \forall t \in (0, T).$$  

(22)

We can specify either $\delta \psi(t)$ to be zero or $\delta p(t)$ to be zero. No other 'upstream' or 'downstream' boundary conditions need to be specified.

It can be shown that the values of the pressure at the corresponding points along I–II and IV–III are related by the following equation:

$$p((x_1)_{IV}, x_2, t) = p((x_1)_I, x_2, t) + \Delta p(t) \quad \forall x_2 \in ((x_2)_I, (x_2)_{II}) \quad \forall t \in (0, T).$$  

(23)

The proof for (23) is very similar to the proof for (16).

From the proofs for (16) and (23) it is clear that the integral of $\partial p/\partial \tau$ along the closed contour I–II–III–IV–I is zero. Therefore for biperiodic flows we need to use only $(q-1)$ of the $q$ equations given by (9) ((10) for inviscid flows). In our implementation we set the value of the stream function along one of the internal boundaries equal to zero and exclude from the set of equations given by (9) ((10) for inviscid flows) the one associated with that internal boundary.
3. The finite element formulation

Let \( \mathcal{E} \) denote the set of elements resulting from the finite element discretization of the computational domain \( \Omega \) into subdomains \( \Omega^e, e = 1, 2, \ldots, n_{el} \), where \( n_{el} \) is the number of elements. Let \( \Gamma^e \) denote the boundary of \( \Omega^e \). We associate to \( \mathcal{E} \) the finite-dimensional space \( H^1 \), where \( H^1 \) is the piecewise bilinear finite element function space.

For uniperiodic flows the trial function spaces for the vorticity and stream function are defined, respectively, as

\[
\mathcal{S}^h = \{ \omega^h | \omega^h \in H^1, \omega^h(x_1, (x_2)_{II}, t) = \omega^h(x_1, (x_2)_I, t) \forall x_1 \in ((x_1)_I, (x_1)_{IV}) \forall t \in (0, T) \}
\]

and

\[
\mathcal{S}^h = \{ \psi^h | \psi^h \in H^1, \psi^h(x_1, (x_2)_{II}, t) = \psi^h(x_1, (x_2)_I, t) + \Delta \psi(t) \forall x_1 \in ((x_1)_I, (x_1)_{IV}) \forall t \in (0, T) \},
\]

\[
\begin{align*}
\partial \psi^h / \partial x_2 &= U_1(t) \text{ on } \Gamma_{01} \times (0, T), \\
\psi^h_{IV} - \psi^h_I &= \delta \psi(t) \text{ on } (0, T), \\
\partial \psi^h / \partial x_2 &= U_1(t) \text{ on } \Gamma_{03} \times (0, T), \\
\partial \psi^h / \partial \tau &= 0 \text{ on } \Gamma_k \times (0, T), k = 1, 2, \ldots, q). \tag{25}
\end{align*}
\]

For biperiodic flows the trial function spaces are defined as

\[
\mathcal{S}^h = \{ \omega^h | \omega^h \in H^1, \omega^h((x_1)_I, (x_2)_I, t) = \omega^h((x_1)_I, (x_2)_I, t) \forall x_1 \in ((x_1)_I, (x_1)_{IV}) \forall x_2 \in ((x_2)_I, (x_2)_{II}) \forall t \in (0, T) \}
\]

and

\[
\begin{align*}
\mathcal{S}^h = \{ \psi^h | \psi^h \in H^1, \psi^h((x_1)_I, (x_2)_I, t) = \psi^h((x_1)_I, (x_2)_I, t) + \Delta \psi(t) \forall x_1 \in ((x_1)_I, (x_1)_{IV}) \forall x_2 \in ((x_2)_I, (x_2)_{II}) \forall t \in (0, T) \},
\end{align*}
\]

\[
\begin{align*}
\partial \psi^h / \partial \tau &= 0 \text{ on } \Gamma_k \times (0, T), k = 1, 2, \ldots, q). \tag{27}
\end{align*}
\]

As it was stated in Section 2, whether the flow is uniperiodic or biperiodic, we can specify either \( \delta \psi(t) \) (i.e., the total downward flow rate between points I and IV) or \( \delta p(t) \) (i.e., the pressure difference between the upper and lower computational boundaries). If we choose to specify \( \delta p(t) \) we need to write a variational formulation to implement this condition. To do this we define a test ‘ring function’ space associated with the boundary \( \Gamma_{01} \). That is,

\[
V^h_{01I} = \{ R^h | R^h \in H^1, R^h|_{\partial \Omega^e} = 0 \forall \Omega^e \in \mathcal{E}_{01}, \partial R^h / \partial \tau = 0 \text{ on } \Gamma_{01} \}, \tag{28}
\]

where \( \mathcal{E}_{01} \) is the set of elements adjacent to the boundary \( \Gamma_{01} \), i.e.,
The variational formulation corresponding to specifying $\delta p(t)$ (equal to zero) can then be written as follows:

Find $\omega^h \in \tilde{S}^h$ and $\psi^h \in S^h$, such that

$$\int_{\Omega_0} \nabla R_h \cdot (\nabla \psi - u \omega^h + \nu \nabla \omega^h) \, d\Omega = 0 \quad \forall R_h \in V_{01R}^h,$$

where a superposed dot denotes the time derivative and $\Omega_0$ is the subset of $\Omega$ spanned by $\mathcal{E}_{01}$.

To derive (30) we first write the momentum equation in $\tau$ direction along the boundary $\Gamma_{01}$.

$$\partial u_\tau / \partial t + \partial (\frac{1}{2} \| u \|^2) / \partial \tau - u_\tau \omega + (1/\rho) \partial p / \partial \tau + \nu \partial \omega / \partial n = 0.$$  

Multiplying (31) with $R_h$ and integrating from I to II we get

$$\int_{I}^{II} R_h (\partial u_\tau / \partial t - u_\tau \omega + \nu \partial \omega / \partial n) \, d\Gamma = 0 \quad \forall R_h \in V_{01R}^h.$$

The integral of the second term in (31) vanishes because of the periodicity of $u$, whereas the integral of the fourth term vanishes because we specify $\delta p(t)$ to be zero. We can also multiply (1) with $R_h$ and integrate over the subdomain $\Omega_{01}$; adding this to (32) we obtain

$$\int_{I}^{II} R_h (\partial u_\tau / \partial t - u_\tau \omega + \nu \partial \omega / \partial n) \, d\Gamma 
+ \int_{\Omega_0} R_h (\omega + \nabla \cdot (u \omega - \nu \nabla \omega)) \, d\Omega = 0 \quad \forall R_h \in V_{01R}^h.$$

By using (2) and (6) we can rewrite (33) as

$$\int_{I}^{II} R_h (\partial \psi / \partial n - u_\tau \omega + \nu \partial \omega / \partial n) \, d\Gamma 
+ \int_{\Omega_0} R_h (-\nabla^2 \psi + \nabla \cdot (u \omega - \nu \nabla \omega)) \, d\Omega = 0 \quad \forall R_h \in V_{01R}^h.$$

If we remember the specific nature of the ‘ring function’ $R_h$ and the periodicity of the flow field, after performing an integration-by-parts on the domain integral, we get (30).

For all other variational formulations we refer the interested reader to [1, 2]; the differences in the remaining variational formulations needed here and the ones given in [1, 2] are quite minor.

In our computations we employ four-node bilinear elements for both the vorticity and the stream function. The time integration is achieved by a block-iteration method (see [1, 2]) which is based on a predictor/multi-corrector scheme and is second-order accurate in time.
4. Numerical examples

All numerical examples considered in this paper involve periodic arrays of staggered cylinders with equal radii (1.0). The differences between these examples are based on the type of periodicity (i.e. uniperiodic or biperiodic), the number of cylinders used in the computational domain, and the arrangement style of the cylinders.

The distance between the centers of neighboring cylinders is always 4.0 in both \( x_1 \) and \( x_2 \) directions. The upper and lower computational boundaries are located at a distance of 4.0 from the centerline of the computational domain. The distance between the upstream computational boundary and the center of the first cylinder is 23.0 for the uniperiodic flows and 2.0 for the biperiodic flows. The distance between the downstream computational boundary and the center of the last cylinder is 63.0 for the uniperiodic flows and 2.0 for the biperiodic flows.

The boundary conditions imposed are as described in Section 2. Unless stated otherwise we set \( \Psi(t) \) equal to zero and for the uniperiodic flows specify the normal derivative of the vorticity at the downstream boundary to be zero. For the uniperiodic flows the magnitude of the uniform upstream velocity is 1.0; the average upstream velocity for the biperiodic flows is also 1.0. The Reynolds number is based on this value of the upstream velocity and the cylinder diameter. We performed computations with Reynolds number 0 (Stokes' flow) and 100.

The initial condition for the uniperiodic flows consists of specifying the vorticity to be zero everywhere; for the biperiodic flows we take the Stokes' solution as the initial condition. Unless specified otherwise the time step size is 0.04.

![Uniperiodic flow with a two-cylinder computational domain: the finite element mesh (5152 elements and 5376 nodes).](image)
Uniperiodic flow with a two-cylinder computational domain

In this problem the computational domain has two cylinders arranged as shown in Fig. 2. The finite element mesh has 5152 elements and 5376 nodes. The lower picture in Fig. 2 depicts the part of the mesh spanning over the $8.0 \times 8.0$ square region around the two cylinders. Similar problems were studied in [6] by taking the horizontal centerline of the upper and lower cylinders as the upper and lower computational boundaries (i.e., by using half of the computational domain employed here) and by assuming symmetry conditions (i.e., zero normal velocity and shear stress) at these boundaries.

Fig. 3. Uniperiodic flow with a two-cylinder computational domain: the vorticity and stream function for the Stokes' problem.
Figure 3 shows the Stokes’ solution for this flow configuration. The lower pictures are obtained by patching together five computational domains.

Figures 4–7 show the time history of the flow field for $Re = 100$ up to $t = 120$. At $t = 8$ (Fig. 4) the flow field is in its relatively early stage; at this stage we observe that the assumption of symmetry conditions along the horizontal centerline of both cylinders is valid. However, at $t = 64$ (Fig. 5) this assumption is definitely invalid. Figure 6 shows how the downstream vortices are formed. Figure 7 shows the flow field at the instant ($t = 120$) we terminated the computation.

Fig. 4. Uniperiodic flow with a two-cylinder computational domain: the vorticity and stream function at $t = 8$. 
For the same problem but with the condition $\delta p(t) = 0$ we were not able to obtain numerical solutions which made physical sense.

**Biperiodic flow with a two-cylinder computational domain**

The finite element mesh used in this problem is exactly the same as the partial mesh depicted by the lower picture in Fig. 2. This mesh has 2048 elements and 2175 nodes.

Figure 8 shows the Stokes' solution for this flow configuration. The lower pictures are obtained by patching together nine computational domains.
Figure 9 shows the flow field for Re = 100 in its early stage \((t = 8)\); at this stage the assumption of symmetry conditions along the horizontal centerlines of the cylinders is valid. In Fig. 10 \((t = 64)\) we see that the symmetry is lost. Figures 11 and 12 show the flow field at \(t = 160\) and 200.

Figures 13–16 show the time history of the flow field for the same problem but with the condition \(\delta p(t) = 0\). The results are quite similar to those obtained with the condition \(\delta \psi(t) = 0\); however, in this case the symmetry with respect to the horizontal centerlines of the cylinders is lost earlier.
Uniperiodic flow with a ten-cylinder computational domain

In this problem the computational domain has ten cylinders arranged as shown in Fig. 17. The finite element mesh has 5300 elements and 5686 nodes. The lower picture in Fig. 17 depicts part of the mesh spanning over $8.0 \times 8.0$ square region around a pair cylinders. The time step size in this problem is 0.05. Similar problems were studied in [4, 5] by taking the horizontal centerline of the upper and lower cylinders as the upper and lower computational boundaries and by assuming symmetry conditions at these boundaries.
Fig. 8. Biperiodic flow with a two-cylinder computational domain: the vorticity and stream function for the Stokes' problem.

Fig. 9. Biperiodic flow with a two-cylinder computational domain: the vorticity and stream function at $t = 8$. 

Fig. 10. Biperiodic flow with a two-cylinder computational domain: the vorticity and stream function at $t = 64$.

Fig. 11. Biperiodic flow with a two-cylinder computational domain: the vorticity and stream function at $t = 160$. 
Fig. 12. Biperiodic flow with a two-cylinder computational domain: the vorticity and stream function at $t = 200$.

Fig. 13. Biperiodic flow with a two-cylinder computational domain: (with the condition $\delta p(t) = 0$): the vorticity and stream function at $t = 8$. 
Fig. 14. Biperiodic flow with a two-cylinder computational domain: (with the condition $\delta p(t) = 0$): the vorticity and stream function at $t = 64$.

Fig. 15. Biperiodic flow with a two-cylinder computational domain: (with the condition $\delta p(t) = 0$): the vorticity and stream function at $t = 160$. 
Fig. 16. Biperiodic flow with a two-cylinder computational domain: (with the condition $\delta p(t) = 0$): the vorticity and stream function at $t = 200$.

Fig. 17. Uniperiodic flow with a ten-cylinder computational domain: the finite element mesh (5300 elements and 5686 nodes).
Fig. 18. Uniperiodic flow with a ten-cylinder computational domain: the vorticity and stream function for the Stokes' problem.

Fig. 19. Uniperiodic flow with a ten-cylinder computational domain: the vorticity and stream function at $t = 10$. 
Fig. 20. Uniperiodic flow with a ten-cylinder computational domain: the vorticity and stream function at $t = 110$.

Fig. 21. Uniperiodic flow with a ten-cylinder computational domain: the vorticity and stream function at $t = 120$. 
Figure 18 shows the Stokes' solution for this flow configuration. The lower pictures are obtained by patching together five computational domains.

Figures 19–22 show the time history of the flow field for $Re = 100$ up to $t = 180$. At $t = 10$ (Fig. 19) the flow field retains the symmetry conditions along the horizontal centerlines of the cylinders. At $t = 110$ (Fig. 20) we observe that the symmetry no longer exists.

Figure 21 shows how the downstream vortices are formed. Figure 22 shows the flow field at the instant ($t = 180$) the computation was terminated.

5. Concluding remarks

We have tested our solution strategies on various uniperiodic and biperiodic flow problems involving arrays of cylinders with Reynolds number 0 (Stokes' flow) and 100. No in-depth physical study was intended in this paper and therefore relatively coarse meshes were employed.

We have shown that in all cases for Reynolds number 100 the solution becomes unsteady and ceases to satisfy the symmetry conditions along the horizontal centerlines of the cylinders.
Therefore, in this type of problems, one should be careful with using assumptions involving such symmetry conditions.

Based on our preliminary results we think all the test problems considered here are interesting and challenging. We plan to subject these problems to a careful physical study using reliable solution techniques free from unrealistic assumptions; we believe we have demonstrated in this paper that we have the computational capability to achieve that.

References