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DCDD in Finite Element Computation of Turbulent Flows

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Abstract: The Streamline-Upwind/Petrov-Galerkin (SUPG) and Pressure-Stabilizing/Petrov-Galerkin (PSPG) methods are among the most popular stabilized formulations in finite element computation of flow problems. The Discontinuity Capturing Directional Dissipation (DCDD) was introduced as a complement to the SUPG and PSPG stabilizations for the computation of incompressible flows in the presence of sharp solution gradients. The DCDD takes effect where there is a sharp gradient in the velocity field and introduces dissipation in the direction of that gradient. We describe how the DCDD, in combination with the SUPG and PSPG stabilizations, can be applied to computation of turbulent flows.

Key words: Stabilized formulations, SUPG and PSPG stabilizations, DCDD stabilization

1 INTRODUCTION

Most finite element techniques reported in the past two decades for flow computations are based on stabilized formulations. The Streamline-Upwind/Petrov-Galerkin (SUPG) formulation for incompressible flows [1], the SUPG formulation for compressible flows [2], and the Pressure-Stabilizing/Petrov-Galerkin (PSPG) formulation for incompressible flows [3] are some of the most prevalent stabilized methods and have well-known advantages. The stabilized formulation introduced in [4] for advection–diffusion–reaction equations included a shock-capturing (discontinuity-capturing) term, and precluded augmentation of the SUPG effect by the discontinuity-capturing effect when the advection and discontinuity directions coincide. A stabilization parameter, "τ", is embedded in the SUPG and PSPG formulations. It involves a measure of the local length scale (also known as "element length") and other parameters such as the element Reynolds and Courant numbers. Various element lengths and τs were proposed starting with those in [1] and [2], followed by the one introduced in [4], and those proposed in the subsequently reported SUPG and PSPG methods. Recently, new ways of computing the τs were introduced in [5, 6] in the context of the advection-diffusion equation and the Navier–Stokes equations of incompressible flows.

To be used in conjunction with the SUPG/PSPG formulation of incompressible flows, a Discontinuity-Capturing Directional Dissipation (DCDD) stabilization was introduced in [7, 6] for computation of flow fields with sharp gradients. This involved a second element length scale, which was also also introduced in [7, 6] and is based on the solution gradient. This new element length scale is used together with the element length scales already defined in [4] and [5]. Recognizing this second element length as a diffusion length scale, new stabilization parameters for the diffusive limit were introduced in [6]. The DCDD takes effect where there is a sharp gradient in the velocity field and introduces dissipation in the direction of that gradient. The way the DCDD is added to the formulation precludes augmentation of the SUPG effect by the DCDD effect when the advection and discontinuity directions coincide. In this paper, we apply the DCDD to computation of turbulent flows in complex wall-bounded flow configurations. In these computations, we use a parallel multi-level method [8]. The parallel solution algorithm includes an overlapping domain decomposition technique based on an inexact explicit nonlinear Schwarz method introduced in [9].

2 GOVERNING EQUATIONS

Let $\Omega \subset I\!R^{n_{sd}}$ be the spatial domain with boundary Γ, and $(0,T)$ be the time domain. The Navier–Stokes equations of incompressible flows can be written on Ω and $\forall t \in (0,T)$ as

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} - \mathbf{f}\right) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} \;= 0, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} \;= 0, \tag{1}$$

where ρ, $\mathbf{u}$ and $\mathbf{f}$ are the density, velocity and the external force, respectively. The stress tensor $\boldsymbol{\sigma}$ is defined as $\boldsymbol{\sigma}(p,\mathbf{u}) = -p\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u})$. Here p is the pressure, $\mathbf{I}$ is the identity tensor, $\mu = \rho\nu$ is the viscosity, ν is the kinematic viscosity, and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the strain-rate tensor: $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left((\boldsymbol{\nabla}\mathbf{u}) + (\boldsymbol{\nabla}\mathbf{u})^T\right)$. The essential and natural boundary conditions for Eq. (1) are represented as $\mathbf{u} = \mathbf{g}$ on $\Gamma_{\rm g}$ and $\mathbf{n}\cdot\boldsymbol{\sigma} = \mathbf{h}$ on $\Gamma_{\rm h}$, where Γ_g and Γ_h are complementary subsets of the boundary Γ, $\mathbf{n}$ is the unit normal vector, and $\mathbf{g}$ and $\mathbf{h}$ are given functions. A divergence-free velocity field $\mathbf{u}_0(\mathbf{x})$ is specified as the initial condition.

3 SUPG/PSPG STABILIZATIONS

We form some suitably-defined finite-dimensional trial solution and test function spaces for velocity and pressure: $\mathcal{S}^h_{\mathbf{u}}$, $\mathcal{V}^h_{\mathbf{u}}$, $\mathcal{S}^h_p$ and $\mathcal{V}^h_p = \mathcal{S}^h_p$. The stabilized finite element formulation is written as follows: find $\mathbf{u}^h \in \mathcal{S}^h_{\mathbf{u}}$ and $p^h \in \mathcal{S}^h_p$ such that $\forall \mathbf{w}^h \in \mathcal{V}^h_{\mathbf{u}}$ and $\forall q^h \in \mathcal{V}^h_p$:

$$\begin{aligned}
& \int_\Omega \mathbf{w}^h\cdot\rho\left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h\cdot\boldsymbol{\nabla}\mathbf{u}^h - \mathbf{f}^h\right) d\Omega + \int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}^h) : \boldsymbol{\sigma}(p^h,\mathbf{u}^h) d\Omega - \int_{\Gamma_{\rm h}} \mathbf{w}^h\cdot\mathbf{h}^h d\Gamma \\
+ \;& \int_\Omega q^h \boldsymbol{\nabla}\cdot\mathbf{u}^h d\Omega + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \frac{1}{\rho}\left[\tau_{\rm SUPG}\rho\mathbf{u}^h\cdot\nabla\mathbf{w}^h + \tau_{\rm PSPG}\nabla q^h\right]\cdot\left[\text{Ł}(p^h,\mathbf{u}^h) - \rho\mathbf{f}^h\right] d\Omega \\
+ \;& \sum_{e=1}^{n_{el}} \int_{\Omega^e} \nu_{\rm LSIC} \boldsymbol{\nabla}\cdot\mathbf{w}^h \rho \boldsymbol{\nabla}\cdot\mathbf{u}^h d\Omega \;=\; 0,
\end{aligned} \tag{2}$$

where

$$\text{Ł}(q^h,\mathbf{w}^h) = \rho\left(\frac{\partial \mathbf{w}^h}{\partial t} + \mathbf{u}^h\cdot\boldsymbol{\nabla}\mathbf{w}^h\right) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(q^h,\mathbf{w}^h). \tag{3}$$

Here $\tau_{\rm SUPG}$, $\tau_{\rm PSPG}$ and $\nu_{\rm LSIC}$ are the SUPG, PSPG and LSIC (least-squares on incompressibility constraint) stabilization parameters.

4 CALCULATION OF THE UGN/RGN-BASED STABILIZATION PARAMETERS

Various ways of calculating the stabilization parameters for incompressible flows were covered earlier in detail in [5, 6, 10, 11]. In this section we focus on the versions of the stabilization parameters (τs) denoted by the subscript $_{\mathrm{UGN}}$, namely the UGN/RGN-based stabilization parameters. For this purpose, we first define the unit vectors $\mathbf{s}$ and $\mathbf{r}$:

$$\mathbf{s} = \frac{\mathbf{u}^h}{\|\mathbf{u}^h\|} \,, \qquad \mathbf{r} = \frac{\boldsymbol{\nabla}\|\mathbf{u}^h\|}{\|\,\boldsymbol{\nabla}\|\mathbf{u}^h\|\,\|} \,. \tag{4}$$

The components of $(\tau_{\mathrm{SUPG}})_{\mathrm{UGN}}$ corresponding to the advection-, transient- and diffusion-dominated limits were defined in [6, 10, 11] as follows:

$$\tau_{\mathrm{SUGN1}} = \left(\sum_{a=1}^{n_{en}} |\mathbf{u}^h \cdot \boldsymbol{\nabla} N_a|\right)^{-1}, \qquad \tau_{\mathrm{SUGN2}} = \frac{\Delta t}{2}, \qquad \tau_{\mathrm{SUGN3}} = \frac{h_{\mathrm{RGN}}^2}{4\nu} \,, \tag{5}$$

where n_{en} is the number of element nodes and N_a is the interpolation function associated with node a, and the "element length" h_{RGN} is defined as

$$h_{\mathrm{RGN}} = 2\left(\sum_{a=1}^{n_{en}} |\mathbf{r} \cdot \boldsymbol{\nabla} N_a|\right)^{-1}. \tag{6}$$

Writing a direct expression for τ_{SUGN1} as given by Eq. (5) was pointed out in [6, 10, 11]. The expression for h_{RGN} as given by Eq. (6) was first introduced in [7, 6]. We now define $(\tau_{\mathrm{SUPG}})_{\mathrm{UGN}}$, $(\tau_{\mathrm{PSPG}})_{\mathrm{UGN}}$, and $(\nu_{\mathrm{LSIC}})_{\mathrm{UGN}}$ as follows:

$$(\tau_{\mathrm{SUPG}})_{\mathrm{UGN}} = \left(\frac{1}{\tau_{\mathrm{SUGN1}}^r} + \frac{1}{\tau_{\mathrm{SUGN2}}^r} + \frac{1}{\tau_{\mathrm{SUGN3}}^r}\right)^{-\frac{1}{r}}, \tag{7}$$

$$(\tau_{\mathrm{PSPG}})_{\mathrm{UGN}} = (\tau_{\mathrm{SUPG}})_{\mathrm{UGN}}, \tag{8}$$

$$(\nu_{\mathrm{LSIC}})_{\mathrm{UGN}} = (\tau_{\mathrm{SUPG}})_{\mathrm{UGN}} \, \|\mathbf{u}^h\|^2 \,. \tag{9}$$

Eq. (7) is based on the inverse of $(\tau_{\mathrm{SUPG}})_{\mathrm{UGN}}$ being defined as the r-norm of the vector with components $\frac{1}{\tau_{\mathrm{SUGN1}}}$, $\frac{1}{\tau_{\mathrm{SUGN2}}}$ and $\frac{1}{\tau_{\mathrm{SUGN3}}}$. We note that the higher the integer r is, the sharper the switching between τ_{SUGN1}, τ_{SUGN2} and τ_{SUGN3} becomes. This "*r-switch*" was introduced in [5]. Typically, $r = 2$. The expressions for τ_{SUGN3} and $(\nu_{\mathrm{LSIC}})_{\mathrm{UGN}}$ were proposed in [6, 10, 11].

5 DISCONTINUITY-CAPTURING DIRECTIONAL DISSIPATION (DCDD)

As an alternative to the LSIC stabilization, the Discontinuity-Capturing Directional Dissipation (DCDD) stabilization was proposed in [7, 6]. In describing the DCDD stabilization, we first define the "DCDD viscosity" ν_{DCDD}:

$$\nu_{\mathrm{DCDD}} = \frac{1}{2}\left(\frac{\|\mathbf{u}^h\|}{u_{\mathrm{ref}}}\right)^2 (h_{\mathrm{DCDD}})^2 \,\|\,\boldsymbol{\nabla}\|\mathbf{u}^h\|\,\| \,. \tag{10}$$

where $h_{\mathrm{DCDD}} = h_{\mathrm{RGN}}$ and u_{ref} is a reference velocity. The DCDD stabilization is defined as

$$S_{\mathrm{DCDD}} = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \rho \boldsymbol{\nabla}\mathbf{w}^h : \left([\nu_{\mathrm{DCDD}}\mathbf{rr} - \boldsymbol{\kappa}_{\mathrm{CORR}}] \cdot \boldsymbol{\nabla}\mathbf{u}^h\right) d\Omega \,, \tag{11}$$

where $\boldsymbol{\kappa}_{\mathrm{CORR}} = \nu_{\mathrm{DCDD}}(\mathbf{r}\cdot\mathbf{s})^2\mathbf{ss}$.

In the LES model we use in our computations, the subgrid viscosity based on the Smagorinsky model is defined as follows: $\nu_{\mathrm{SMAG}} = (l_{\mathrm{SMAG}})^2 \,\left(2\,\boldsymbol{\varepsilon}(\mathbf{u}^h) : \boldsymbol{\varepsilon}(\mathbf{u}^h)\right)^{\frac{1}{2}}$. Here l_{SMAG} is the Smagorinsky length scale defined as $l_{\mathrm{SMAG}} = (C_{\mathrm{SMAG}}\,\Delta_{\mathrm{SMAG}})$, where C_{SMAG} ranges between 0.1

and 0.17, and Δ_{SMAG} is the grid filter width: $\Delta_{\text{SMAG}} = 2\ (\Delta x\ \Delta y\ \Delta z)^{\frac{1}{3}}$. To take into account the near-wall effects, the Smagorinsky length scale is modified by using a Van Driest damping function as follows: $l_{\text{SMAG}} = (C_{\text{SMAG}}\ \Delta_{\text{SMAG}}) \left[1 - \mathrm{e}^{\left(\frac{y^+}{A^+}\right)}\right]$, where A^+ is a constant set to 26, and $y^+ = \frac{y u_\tau}{\nu}$. Here u_τ is the friction velocity: $u_\tau = \left(\frac{\tau_w}{\rho}\right)^{\frac{1}{2}}$, where τ_w is the wall shear stress.

6 TEST COMPUTATION

This is a fully developed 3D plane channel flow involving a unidirectional flow configuration characterized by a friction Reynolds number of $Re_\tau = u_\tau \delta/\nu = 395$, where δ is the half channel width. The direct numerical simulation (DNS) of this problem reported in [12] is regarded as highly accurate due to the very fine discretization used. Here we use that as benchmark data. The problem geometry is shown in Figure 1. The inflow boundary conditions are extracted

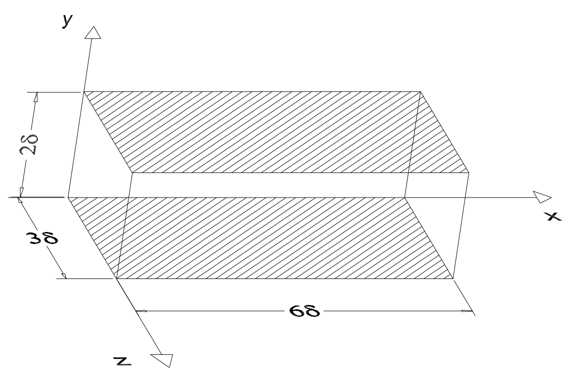

Figure 1: Plane channel flow at $Re_\tau = 395$. Problem geometry.

from a developed flow field computed with a RANS model, plus fluctuations in the form of a Gauss random distribution function in time. No-slip conditions are imposed on the walls, and the periodicity conditions at the lateral boundaries normal to the z axis. The domain is discretized with a structured grid consisting of $120 \times 56 \times 56$ Q1Q1 elements. The grid is more refined near the walls (in the y direction) and also near the lateral boundaries normal to the z axis. The grid is uniform in the x direction. At the wall $\Delta y^+ = 1.8$, while $\Delta x^+ = 65$ and $\Delta z^+ = 21$. In the direction normal to the wall the first four nodes are in the region $y^+ \leq 10$. The CFL number is 0.5. In the LES computation, we set $C_{\text{SMAG}} = 0.15$. The problem is computed for 900 time steps to develop into a fully turbulent flow, and then an additional 6600 time steps to gather statistics.

Figures 2 and 3 shows the Reynolds stress components uu^+ $(=uu/u_\tau^2)$ and vv^+ $(=vv/v_\tau^2)$ obtained with the DNS simulation, LES model, and DCDD stabilization. We observe that in the LES and DCCD computations the uu^+ profile is predicted well within the viscous sub-layer region up to the log-layer region. We also observe that the LES model and DCCD stabilization overpredict the DNS stress in regions approaching the channel centerline. For the Reynolds stress component vv^+, which is critical in transitional flows, we see in Figure 3 that the results obtained with the DNS simulation, LES model, and DCDD stabilization compare well.

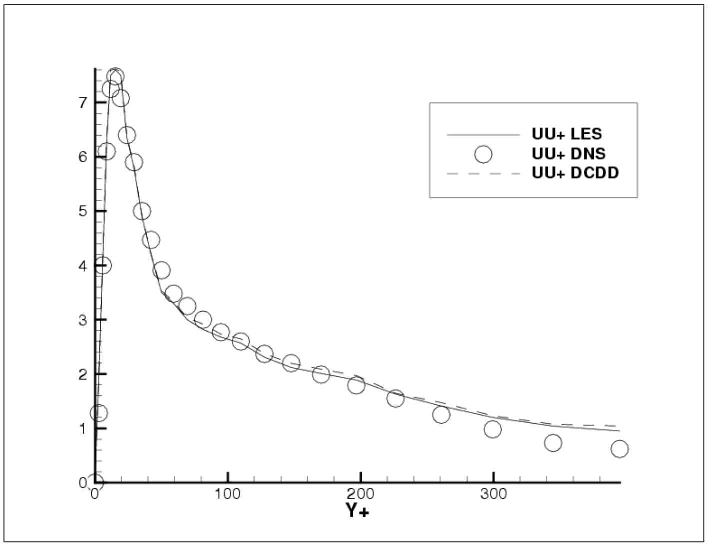

Figure 2: Plane channel flow at $Re_\tau = 395$. Reynolds stress component uu^+.

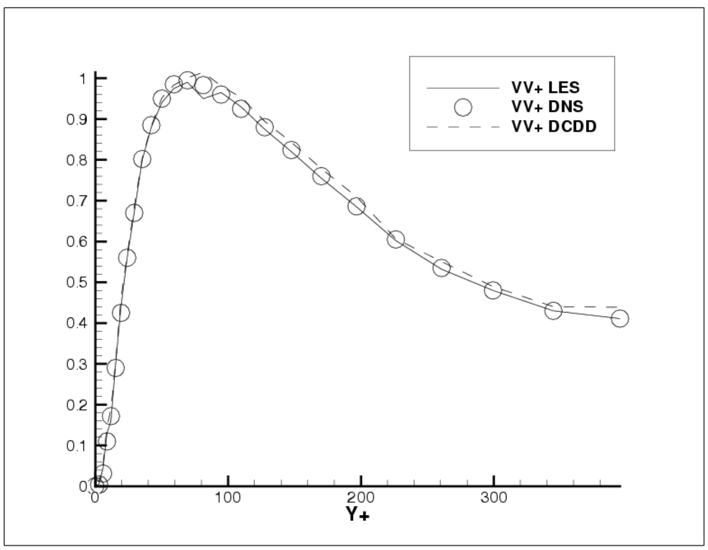

Figure 3: Plane channel flow at $Re_\tau = 395$. Reynolds stress component vv^+.

7 CONCLUDING REMARKS

We described how the DCDD stabilization, in conjunction with the SUPG and PSPG stabilizations, can be applied to the computation of turbulent flows. The DCDD takes effect where there is a sharp gradient in the velocity field. The formulation precludes augmentation of the SUPG effect by the DCDD effect when the advection and discontinuity directions coincide. With a test computation, we compared the results obtained with the DCDD stabilization with those obtained with an LES turbulence model. The comparison shows that the DCDD stabilization has a good potential in computation of turbulent flows.

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