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ADAPTIVE DETERMINATION OF THE FINITE ELEMENT STABILIZATION PARAMETERS

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Abstract. *We propose adaptive determination of the stabilization parameters used in the stabilized finite element methods such as the streamline-upwind/Petrov-Galerkin (SUPG) and pressure-stabilizing/Petrov-Galerkin (PSPG) formulations. The parameters are calculated based on the element-level matrices and vectors, which automatically take into account the local length scales, advection field and the Reynolds number. We describe how we determine these parameters in the context of first a time-dependent advection-diffusion equation and then the Navier-Stokes equations of unsteady incompressible flows.*

1 INTRODUCTION

Stabilized formulations played a major role in the past two decades in making the finite element method a reliable and powerful approach in flow simulation and modeling. Among the most notable stabilized formulations are the streamline-upwind/Petrov-Galerkin (SUPG) formulation for incompressible flows (Brooks and Hughes [1]), SUPG formulation for compressible flows (Tezduyar and Hughes [2]), Galerkin/least-squares (GLS) formulation (Hughes et al. [3]), and pressure-stabilizing/Petrov-Galerkin (PSPG) formulation for incompressible flows (Tezduyar [4]). These stabilization techniques prevent numerical oscillations and other instabilities in solving problems with high Reynolds and/or Mach numbers and shocks and strong boundary layers, as well as when using equal-order interpolation functions for velocity and pressure and other unknowns. The SUPG, GLS and PSPG formulations stabilize the method without introducing excessive numerical dissipation. Because its symptoms are not necessarily qualitative, excessive numerical dissipation is not always easy to detect. This concern makes it desirable to seek and employ stabilized formulations developed with objectives that include keeping numerical dissipation to a minimum.

When the implementation of the SUPG, GLS or PSPG formulations is based on a sound understanding of these methods, they perform quite well. Furthermore, this class of stabilized formulations substantially improve the convergence rate in iterative solution of the large, coupled nonlinear equation system that needs to be solved at every time step of a flow computation. Such nonlinear systems are typically solved with the Newton-Raphson method, which involves, at its every iteration step, solution of a large, coupled linear equation system. It is in iterative solution of such linear equation systems that using a good stabilized method makes substantial difference in convergence, and this was pointed out in Tezduyar et al. [5].

The SUPG formulation for incompressible flows was introduces in Hughes and Brooks [6], with detailed description of the formulation and numerical examples given in Brooks and Hughes [1]. The SUPG formulation for compressible flows was first introduced, in the context of conservation variables, in Tezduyar and Hughes [2]. After that, several SUPG-like methods for compressible flows were developed. Taylor-Galerkin method (Donea [7]), for example, is very similar, and under certain conditions is identical, to one of the stabilization methods introduced in Tezduyar and Hughes [2]. Another example of the subsequent SUPG-like methods for compressible flows in conservation variables is the streamline-diffusion method described in Johnson et al. [8]. Later, following Tezduyar and Hughes [2], the SUPG formulation for compressible flows was recast in entropy variables and supplemented with a shock-capturing term (Hughes et al. [9]). It was shown in Le Beau and Tezduyar [10] and Aliabadi et al. [11] that, the SUPG formulation introduced in Tezduyar and Hughes [2], when supplemented with a similar shock-capturing term, is very comparable in accuracy to the one that was recast in entropy variables. It was shown in Le Beau et al. [12] for inviscid flows and in Aliabadi et al. [11] for viscous flows that for

2D test problems computed, the SUPG formulation in conservation and entropy variables yield indistinguishable results. A recently introduced SUPG formulation for compressible flows in augmented conservation variables (Mittal and Tezduyar [13]) leads to a proper incompressible flow formulation in the limit as the Mach number is taken to zero.

In the SUPG, GLS and PSPG methods, selection of the stabilization parameter, which is almost universally known as "τ", has attracted a significant amount of attention and research. This stabilization parameter involves a measure of the local length scale (also known as "element length") and other parameters such as the local Reynolds and Courant numbers. Selection of the "element length" also attracted attention. Various "element length"s and "τ"s were proposed starting with those in Brooks and Hughes [1] and Tezduyar and Hughes [2] followed by the one introduced in Tezduyar and Park [14], and those proposed in the subsequently reported SUPG, GLS and PSPG methods. A number of "τ"s, dependent upon spatial and temporal discretizations, were introduced and tested in Tezduyar and Ganjoo [15]. More recently, "τ"s which are applicable to higher-order elements were proposed by Franca et al. [16].

In this paper we describe how we adaptively determine the stabilization parameter "τ". The parameters we propose are calculated from the element-level matrices and vectors, and these automatically take into account the local length scales as well as the advection field and the element-level Reynolds number. In Section 2, we describe the calculation of the stabilization parameter for a time-dependent advection-diffusion equation, and in Section 3 for the Navier-Stokes equations of unsteady incompressible flows. Concluding remarks are given in Section 4.

2 ADVECTION-DIFFUSION EQUATION

Let us consider over a domain Ω with boundary Γ the following time-dependent advection-diffusion equation:

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\phi - \boldsymbol{\nabla} \cdot (\nu \boldsymbol{\nabla}\phi) = 0 \text{ on } \Omega, \tag{1}$$

where ϕ represents the quantity being transported (e.g., temperature, concentration), $\mathbf{u}$ is a divergence-free advection field, and ν is the diffusivity. The essential and natural boundary conditions associated with Eq. (1) are represented as

$$\phi = g \qquad \text{on } \Gamma_g, \tag{2}$$

$$\mathbf{n} \cdot \nu \boldsymbol{\nabla}\phi = h \qquad \text{on } \Gamma_h, \tag{3}$$

where g and h are given functions, $\mathbf{n}$ is the unit normal vector at the boundary, and Γ_g and Γ_h are the complementary subsets of Γ. The initial condition consists of the form

$$\phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}) \text{ on } \Omega. \tag{4}$$

Let us assume that we have constructed some suitably-defined finite-dimensional trial solution and test function spaces $\mathcal{S}_\phi^h$ and $\mathcal{V}_\phi^h$. The stabilized finite element formulation of

Eq. (1) can then be written as follows: find $\phi^h \in \mathcal{S}_\phi^h$ such that $\forall w^h \in \mathcal{V}_\phi^h$:

$$\begin{aligned} & \int_\Omega w^h \left(\frac{\partial \phi^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \phi^h \right) d\Omega + \int_\Omega \boldsymbol{\nabla} w^h \cdot \nu \boldsymbol{\nabla} \phi^h d\Omega \\ + & \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\text{SUPG}} \mathbf{u}^h \cdot \boldsymbol{\nabla} w^h \left(\frac{\partial \phi^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \phi^h - \boldsymbol{\nabla} \cdot \left(\nu \boldsymbol{\nabla} \phi^h \right) \right) d\Omega \\ = & \int_{\Gamma_h} w^h h d\Gamma. \end{aligned} \tag{5}$$

Here n_{el} is the number of elements and Ω^e is the element domain corresponding to element e. τ_{SUPG} is the SUPG (streamline-upwind/Petrov-Galerkin) stabilization parameter.

Let us use the notation $\mathbf{b} : \int_{\Omega^e} (\ldots) d\Omega : \mathbf{b}_\text{V}$ to denote the element-level matrix $\mathbf{b}$ and element-level vector $\mathbf{b}_\text{V}$ corresponding to the element-level integration term $\int_{\Omega^e} (\ldots) d\Omega$. We now define the following element-level matrices and vectors:

$$\mathbf{m} : \quad \int_{\Omega^e} w^h \frac{\partial \phi^h}{\partial t} d\Omega \quad : \mathbf{m}_\text{V}, \tag{6}$$

$$\mathbf{c} : \quad \int_{\Omega^e} w^h \mathbf{u}^h \cdot \boldsymbol{\nabla} \phi^h d\Omega \quad : \mathbf{c}_\text{V}, \tag{7}$$

$$\mathbf{k} : \quad \int_{\Omega^e} \boldsymbol{\nabla} w^h \cdot \nu \boldsymbol{\nabla} \phi^h d\Omega \quad : \mathbf{k}_\text{V}, \tag{8}$$

$$\tilde{\mathbf{k}} : \quad \int_{\Omega^e} \mathbf{u}^h \cdot \boldsymbol{\nabla} w^h \ \mathbf{u}^h \cdot \boldsymbol{\nabla} \phi^h d\Omega \quad : \tilde{\mathbf{k}}_\text{V}, \tag{9}$$

$$\tilde{\mathbf{c}} : \quad \int_{\Omega^e} \mathbf{u}^h \cdot \boldsymbol{\nabla} w^h \frac{\partial \phi^h}{\partial t} d\Omega \quad : \tilde{\mathbf{c}}_\text{V}. \tag{10}$$

We define the element-level Reynolds and Courant numbers as follows:

$$Re = \frac{\|\mathbf{u}^h\|^2}{\nu} \frac{\|\mathbf{c}\|}{\|\tilde{\mathbf{k}}\|}, \tag{11}$$

$$Cr_u = \frac{\Delta t}{2} \frac{\|\mathbf{c}\|}{\|\mathbf{m}\|}, \tag{12}$$

$$Cr_\nu = \frac{\Delta t}{2} \frac{\|\mathbf{k}\|}{\|\mathbf{m}\|}, \tag{13}$$

$$Cr_{\tilde{\nu}} = \frac{\Delta t}{2} \tau_{\text{SUPG}} \frac{\|\tilde{\mathbf{k}}\|}{\|\mathbf{m}\|}, \tag{14}$$

where $\|\mathbf{b}\|$ is the norm of matrix $\mathbf{b}$.

Remark 1 *The Courant numbers defined above can be used for determining the time step size of the computation.*

The components of element-matrix-based τ_{SUPG} are defined as follows:

$$\tau_{\text{S1}} = \frac{\|\mathbf{c}\|}{\|\tilde{\mathbf{k}}\|}, \tag{15}$$

$$\tau_{\text{S2}} = \frac{\Delta t}{2}\frac{\|\mathbf{c}\|}{\|\tilde{\mathbf{c}}\|}, \tag{16}$$

$$\tau_{\text{S3}} = \tau_{\text{S1}} Re = \left(\frac{\|\mathbf{c}\|}{\|\tilde{\mathbf{k}}\|}\right) Re. \tag{17}$$

Remark 2 *Because $\tilde{\mathbf{c}} = \mathbf{c}^T$, for some definitions of the matrix norm, $\|\tilde{\mathbf{c}}\| = \|\mathbf{c}\|$, and therefore $\tau_{S2} = \left(\frac{\Delta t}{2}\right)$.*

Remark 3 *In the special case of a 1D problem, $\tau_{S1} = \left(\frac{h}{2|u|}\right), \tau_{S2} = \left(\frac{\Delta t}{2}\right)$ and $\tau_{S3} = \left(\frac{h^2}{4\nu}\right)$, which are the popular limits for τ_{SUPG} for the advection-dominated, transient-dominated and diffusion-dominated cases, respectively.*

Several different but similar ways have been used to construct τ_{SUPG} from its components. We propose the form

$$\tau_{\text{SUPG}} = \left(\frac{1}{\tau_{\text{S1}}^r} + \frac{1}{\tau_{\text{S2}}^r} + \frac{1}{\tau_{\text{S3}}^r}\right)^{-\frac{1}{r}}, \tag{18}$$

which is based on the inverse of τ_{SUPG} being defined as the r-norm of the vector with components $\frac{1}{\tau_{\text{S1}}}$, $\frac{1}{\tau_{\text{S2}}}$ and $\frac{1}{\tau_{\text{S3}}}$. We note that the higher the integer r is, the sharper the switching between τ_{S1}, τ_{S2} and τ_{S3} becomes.

Remark 4 *It is conceivable that we calculate a separate τ for each element node, or degree of freedom, or element equation. In that case, each component of τ would be calculated separately for each element node, or degree of freedom, or element equation. For this, we first represent an element matrix $\mathbf{b}$ in terms of its row vectors or row matrices: $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n_{ex}}$. If we want a separate τ for each element node, then $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n_{ex}}$, would be the row matrices corresponding to each element node, with $n_{ex} = n_{en}$, where n_{en} is the number of element nodes. If we want a separate τ for each degree of freedom, then $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n_{ex}}$, would be the row matrices corresponding to each degree of freedom, with $n_{ex} = n_{dof}$, where n_{dof} is the number of degrees of freedom. If we want a separate τ for each element equation, then $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n_{ex}}$ would be the row vectors corresponding to each element equation, with $n_{ex} = n_{ee}$, where n_{ee} is the number of element equations. Based on this, the components of τ would be calculated using the norms of these row matrices or vectors, instead of the element matrices. For example, a separate τ_{S1} for each element node would be calculated by using the expression $(\tau_{S1})_a = \frac{\|\mathbf{c}_a\|}{\|\tilde{\mathbf{k}}_a\|}$, $a = 1, 2, \ldots, n_{en}$. We should also note that in some special cases some of these alternative ways of computing τ might give the same result.*

The components of the element-vector-based τ_{SUPG} are defined as follows:

$$\tau_{\text{SV1}} = \frac{\|\mathbf{c}_{\text{V}}\|}{\|\tilde{\mathbf{k}}_{\text{V}}\|}, \tag{19}$$

$$\tau_{\text{SV2}} = \frac{\|\mathbf{c}_{\text{V}}\|}{\|\tilde{\mathbf{c}}_{\text{V}}\|}, \tag{20}$$

$$\tau_{\text{SV3}} = \tau_{\text{SV1}} Re = \left(\frac{\|\mathbf{c}_{\text{V}}\|}{\|\tilde{\mathbf{k}}_{\text{V}}\|} \right) Re. \tag{21}$$

With these three components,

$$(\tau_{\text{SUPG}})_{\text{V}} = \left(\frac{1}{\tau_{\text{SV1}}^r} + \frac{1}{\tau_{\text{SV2}}^r} + \frac{1}{\tau_{\text{SV3}}^r} \right)^{-\frac{1}{r}}. \tag{22}$$

Remark 5 *The definition of τ_{SUPG} given by Eq. (22) can be seen as a nonlinear definition because it depends on the solution. However, in marching from time level n to $n+1$ the element vectors can be evaluated at level n. This might be preferable in some cases, as it spares us from ending up with a nonlinear semi-discrete equation system.*

Remark 6 *In some cases it might be desirable to have a dynamic switching between τ_{SUPG} and $(\tau_{SUPG})_V$ during the computation.*

Remark 7 *Both definitions of τ_{SUPG} are applicable to higher-order elements.*

Remark 8 *It was pointed out in [5], and it is now well-known, that the stabilization substantially improves the convergence in iterative solution of the linear equation system that needs to be solved at each Newton-Raphson step of the solution of the nonlinear equation system encountered at each time step.*

Remark 9 *It is also well-known that using higher-order elements degrades the convergence in iterative solution of such linear equation systems. It has been observed that [17] using interpolation functions which are spatially discontinuous across element boundaries improves the convergence for higher-order elements. Leaving aside the fact that such discontinuous methods come with substantial increases in computational cost which might render the approach impractical in large-scale 3D computations, it is our opinion that this convergence improvement is due to breaking the global, unintended approximation of the lower-order functions by the higher-order functions. As one uses higher- and higher-order functions, the global approximation of the lower-order functions by the higher-order functions gets better, and the approximate and unintended linear dependence of the lower-order functions on the higher-order functions increases. The spatial discontinuity breaks this global approximate linear dependence.*

Remark 10 *In Eq. (5), the SUPG stabilization term involves the residual of the governing equation, which includes the second-order term* $\nabla \cdot (\nu \nabla \phi^h)$*. For linear (triangular and tetrahedral) elements this term vanishes. For bilinear (quadrilateral) and trilinear (hexahedral) elements, this term vanishes for certain special geometries (e.g. rectangles and bricks), and is largely under-represented for more general geometries. This steals away from the consistency of the stabilized finite element formulation as defined by Eq. (5). If one desires to remedy the situation, it is our opinion that this could be accomplished by modifying the stabilized formulation given by Eq. (5) in such a way that the terms representing the discontinuity of the flux* $\nu \nabla \phi^h$ *across element boundaries (i.e. flux "jump" terms) are included in the SUPG stabilization terms. It is also our opinion that the best way to derive effective stabilized formulations is to start with the Galerkin formulation of the problem, reverse integrate-by-parts to derive the finite-dimensional Euler-Lagrange form of the equations, and consider including as a factor in the stabilization terms any of the terms appearing in the Euler-Lagrange form (such as the governing partial differential equations and the flux jump terms as well as the difference between the natural boundary condition and the flux at that boundary). The procedure is described in more detail in Appendix I.*

3 NAVIER-STOKES EQUATIONS OF INCOMPRESSIBLE FLOWS

We write the Navier-Stokes equations of incompressible flows as

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{f}\right) - \nabla \cdot \boldsymbol{\sigma} = 0 \qquad \text{on } \Omega, \tag{23}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{on } \Omega, \tag{24}$$

where ρ is density (constant in this case), $\mathbf{u}$ is the velocity vector, $\mathbf{f}$ is the external force and $\boldsymbol{\sigma}$ is the stress tensor:

$$\boldsymbol{\sigma}(p, \mathbf{u}) = -p\mathbf{I} + \mathbf{T}. \tag{25}$$

Here p is the pressure, $\mathbf{I}$ is the identity tensor, and

$$\mathbf{T} = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}), \tag{26}$$

where $\mu = \rho\nu$ is the viscosity, ν is the kinematic viscosity, and $\boldsymbol{\varepsilon}$ is the strain-rate tensor:

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}((\nabla \mathbf{u}) + (\nabla \mathbf{u})^T). \tag{27}$$

The essential and natural boundary conditions associated with Eq. (23) are represented as

$$\mathbf{u} = \mathbf{g} \text{ on } \Gamma_{\mathrm{g}}, \tag{28}$$

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{h} \text{ on } \Gamma_{\mathrm{h}}. \tag{29}$$

The initial condition on $\mathbf{u}$ is given as

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{30}$$

where $\boldsymbol{\nabla} \cdot \mathbf{u}_0 = 0$.

Let us again assume that we have some suitably-defined finite-dimensional trial solution and test function spaces for velocity and pressure: $\mathcal{S}^h_{\mathbf{u}}$, $\mathcal{V}^h_{\mathbf{u}}$, $\mathcal{S}^h_p$ and $\mathcal{V}^h_p = \mathcal{S}^h_p$. The stabilized finite element formulation of Eqs. (23)-(24) can then be written as follows: find $\mathbf{u}^h \in \mathcal{S}^h_{\mathbf{u}}$ and $p^h \in \mathcal{S}^h_p$ such that $\forall \mathbf{w}^h \in \mathcal{V}^h_{\mathbf{u}}$ and $q^h \in \mathcal{V}^h_p$:

$$\begin{aligned}
& \int_\Omega \mathbf{w}^h \cdot \rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h - \mathbf{f} \right) d\Omega + \int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}^h) : \boldsymbol{\sigma}(p^h, \mathbf{u}^h) d\Omega \\
+ & \int_\Omega q^h \boldsymbol{\nabla} \cdot \mathbf{u}^h d\Omega + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \frac{1}{\rho} [\tau_{\text{SUPG}} \rho \mathbf{u}^h \cdot \nabla \mathbf{w}^h + \tau_{\text{PSPG}} \nabla q^h] \cdot \\
& \left[\rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h \right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(p^h, \mathbf{u}^h) - \rho \mathbf{f} \right] d\Omega \\
+ & \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\text{LSIC}} \boldsymbol{\nabla} \cdot \mathbf{w}^h \rho \boldsymbol{\nabla} \cdot \mathbf{u}^h d\Omega \\
= & \int_{\Gamma_h} \mathbf{w}^h \cdot \mathbf{h}^h d\Gamma.
\end{aligned} \tag{31}$$

Here τ_{PSPG} is the PSPG (pressure-stabilizing/Petrov-Galerkin) stabilization parameter and τ_{LSIC} is the LSIC (least-squares on incompressibility constraint) stabilization parameter.

We now define the following element-level matrices and vectors:

$$\mathbf{m} : \quad \int_{\Omega^e} \mathbf{w}^h \cdot \rho \frac{\partial \mathbf{u}^h}{\partial t} d\Omega \quad : \mathbf{m}_{\text{V}}, \tag{32}$$

$$\mathbf{c} : \quad \int_{\Omega^e} \mathbf{w}^h \cdot \rho (\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h) d\Omega \quad : \mathbf{c}_{\text{V}}, \tag{33}$$

$$\mathbf{k} : \quad \int_{\Omega^e} \boldsymbol{\varepsilon}(\mathbf{w}^h) : 2\mu \boldsymbol{\varepsilon}(\mathbf{u}^h) d\Omega \quad : \mathbf{k}_{\text{V}}, \tag{34}$$

$$\mathbf{g} : \quad \int_{\Omega^e} (\boldsymbol{\nabla} \cdot \mathbf{w}^h) p^h d\Omega \quad : \mathbf{g}_{\text{V}}, \tag{35}$$

$$\mathbf{g}^{\text{T}} : \quad \int_{\Omega^e} q^h (\boldsymbol{\nabla} \cdot \mathbf{u}^h) d\Omega \quad : \mathbf{g}^{\text{T}}_{\text{V}}, \tag{36}$$

$$\tilde{\mathbf{k}} : \quad \int_{\Omega^e} (\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{w}^h) \cdot \rho (\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h) d\Omega \quad : \tilde{\mathbf{k}}_{\text{V}}, \tag{37}$$

$$\tilde{\mathbf{c}} : \quad \int_{\Omega^e} (\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{w}^h) \cdot \rho \frac{\partial \mathbf{u}^h}{\partial t} d\Omega \quad : \tilde{\mathbf{c}}_{\text{V}}, \tag{38}$$

$$\tilde{\gamma} : \quad \int_{\Omega^e} (\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{w}^h) \cdot \boldsymbol{\nabla} p^h d\Omega \quad : \tilde{\gamma}_{\text{V}}, \tag{39}$$

$$\beta : \quad \int_{\Omega^e} \boldsymbol{\nabla} q^h \cdot \frac{\partial \mathbf{u}^h}{\partial t} d\Omega \quad : \beta_{\text{V}}, \tag{40}$$

$$\gamma: \qquad \int_{\Omega^e} \boldsymbol{\nabla} q^h \cdot (\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{u}^h) d\Omega \qquad : \gamma_{\mathrm{V}}, \tag{41}$$

$$\theta: \qquad \int_{\Omega^e} \boldsymbol{\nabla} q^h \cdot \boldsymbol{\nabla} p^h d\Omega \qquad : \theta_{\mathrm{V}}, \tag{42}$$

$$\mathbf{e}: \qquad \int_{\Omega^e} (\boldsymbol{\nabla} \cdot \mathbf{w}^h) \rho (\boldsymbol{\nabla} \cdot \mathbf{u}^h) d\Omega \qquad : \mathbf{e}_{\mathrm{V}}. \tag{43}$$

Remark 11 *In the definition of the element-level matrices listed above, we assume that $\mathbf{u}^h$ appearing in the advective operator (i.e. in $\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{u}^h$ and $\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{w}^h$) is evaluated at time level n rather than $n+1$. The definition would essentially be the same if we, alternatively, assumed that it is evaluated at time level $n+1$ but nonlinear iteration level i rather than $i+1$. Except, in the first option, in the advective operator we use $(\mathbf{u}^h)_n$, whereas in the second option we use $(\mathbf{u}^h)_{n+1}^i$. The second option can be seen as a nonlinear definition. The first option might be preferable in some cases, as it spares us from another level of nonlinearity coming from the way τ is defined. In the definition of the element-level-vectors, we face the same choices in terms of the evaluation of $\mathbf{u}^h$ in the advective operator.*

Remark 12 *We note that $\tilde{\mathbf{c}} = \mathbf{c}^T$ and $\tilde{\gamma} = \gamma^T$.*

The element-level Reynolds and Courant numbers are defined the same way as they were defined before, as given by Eqs. (11)-(14).

Remark 13 *Remark 1 applies also in this case.*

The components of the element-matrix-based τ_{SUPG} are defined the same way as they were defined before, as given by Eqs. (15)-(17).

Remark 14 *Remarks 2 and 3 also apply in this case.*

τ_{SUPG} is constructed from its components the same way as it was constructed before, as give by Eq. (18).

Remark 15 *Remark 4 applies also in this case.*

The components of the element-vector-based τ_{SUPG} are defined the same way as they were defined before, as given by Eqs. (19)-(21). The construction of $(\tau_{\mathrm{SUPG}})_{\mathrm{V}}$ is also the same as it was before, given by Eq. (22).

Remark 16 *Remarks 5 - 7 apply also in this case.*

The components of the element-matrix-based $\tau_{\rm PSPG}$ are defined as follows:

$$\tau_{\rm P1} = \frac{\|\mathbf{g}^{\rm T}\|}{\|\gamma\|}, \tag{44}$$

$$\tau_{\rm P2} = \frac{\Delta t}{2} \frac{\|\mathbf{g}^{\rm T}\|}{\|\beta\|}, \tag{45}$$

$$\tau_{\rm P3} = \tau_{\rm P1} Re = \left(\frac{\|\mathbf{g}^{\rm T}\|}{\|\gamma\|}\right) Re. \tag{46}$$

Remark 17 *Remark 3 applies also in this case.*

$\tau_{\rm PSPG}$ is constructed from its components as follows:

$$\tau_{\rm PSPG} = \left(\frac{1}{\tau_{\rm P1}^r} + \frac{1}{\tau_{\rm P2}^r} + \frac{1}{\tau_{\rm P3}^r}\right)^{-\frac{1}{r}}. \tag{47}$$

Remark 18 *Remark 4 applies also in this case.*

The components of the element-vector-based $\tau_{\rm PSPG}$ are defined as follows:

$$\tau_{\rm PV1} = \tau_{\rm P1}, \tag{48}$$

$$\tau_{\rm PV2} = \tau_{\rm PV1} \frac{\|\gamma_{\rm V}\|}{\|\beta_{\rm V}\|}, \tag{49}$$

$$\tau_{\rm PV3} = \tau_{\rm PV1} Re. \tag{50}$$

With these components,

$$(\tau_{\rm PSPG})_{\rm V} = \left(\frac{1}{\tau_{\rm PV1}^r} + \frac{1}{\tau_{\rm PV2}^r} + \frac{1}{\tau_{\rm PV3}^r}\right)^{-\frac{1}{r}}. \tag{51}$$

Remark 19 *Remarks 5-7 apply also in this case.*

The element-matrix-based $\tau_{\rm LSIC}$ is defined as follows:

$$\tau_{\rm LSIC} = \frac{\|\mathbf{c}\|}{\|\mathbf{e}\|}. \tag{52}$$

Remark 20 *In the special case of a 1D problem,* $\tau_{LSIC} = |u|h/2$.

Remark 21 *Remark 4 applies also in this case.*

We define the element-vector-based $\tau_{\rm LSIC}$ to be identical to the element-matrix-based $\tau_{\rm LSIC}$:

$$(\tau_{\rm LSIC})_{\rm V} = \tau_{\rm LSIC}. \tag{53}$$

Remark 22 *Remark 7 applies also in this case.*

Remark 23 *Remarks 8-9 apply also to the Navier-Stokes equations of incompressible flows.*

Remark 24 *Remark 10 applies also to the stabilized formulation of the Navier-Stokes equations of incompressible flows. However it needs to be pointed out that, for both the advection-diffusion and Navier-Stokes equations, under-representation of the second-order terms* $\boldsymbol{\nabla} \cdot (\nu \boldsymbol{\nabla} \phi^h)$ *and* $\boldsymbol{\nabla} \cdot (2\mu \boldsymbol{\varepsilon}(\mathbf{u}^h))$ *does not steal much from the consistency of the stabilized finite element formulation at Reynolds numbers high enough (i.e.* $Re >> 1$*) to render these second-order terms negligible compared to the advective term.*

For the purpose of comparison with the stabilization parameters we used earlier, we define here those stabilization parameters which are based on an earlier definition of the length scale h [14]:

$$h_{\mathrm{UGN}} = 2\, \|\mathbf{u}^h\| \left(\sum_{a=1}^{n_{en}} |\mathbf{u}^h \cdot \boldsymbol{\nabla} N_a| \right)^{-1}, \tag{54}$$

where N_a is the interpolation function associated with node a. The stabilization parameters are defined as follows:

$$\begin{aligned}
\tau_{\mathrm{SUGN1}} &= \frac{h_{\mathrm{UGN}}}{2\|\mathbf{u}^h\|}, &(55)\\
\tau_{\mathrm{SUGN2}} &= \frac{\Delta t}{2}, &(56)\\
\tau_{\mathrm{SUGN3}} &= \frac{h_{\mathrm{UGN}}^2}{4\nu}, &(57)\\
(\tau_{\mathrm{SUPG}})_{\mathrm{UGN}} &= \left(\frac{1}{\tau_{\mathrm{SUGN1}}^2} + \frac{1}{\tau_{\mathrm{SUGN2}}^2} + \frac{1}{\tau_{\mathrm{SUGN3}}^2} \right)^{-\frac{1}{2}}, &(58)\\
(\tau_{\mathrm{PSPG}})_{\mathrm{UGN}} &= (\tau_{\mathrm{SUPG}})_{\mathrm{UGN}}, &(59)\\
(\tau_{\mathrm{LSIC}})_{\mathrm{UGN}} &= \frac{h_{\mathrm{UGN}}}{2} \|\mathbf{u}^h\|\, z. &(60)
\end{aligned}$$

Here z is given as follows:

$$z = \begin{cases} \left(\frac{Re_{\mathrm{UGN}}}{3}\right) & Re_{\mathrm{UGN}} \le 3, \\ 1 & Re_{\mathrm{UGN}} > 3, \end{cases} \tag{61}$$

where $Re_{\mathrm{UGN}} = \frac{\|\mathbf{u}^h\| h_{\mathrm{UGN}}}{2\nu}$.

Comparisons between the performances of these earlier stabilization parameters and the ones proposed here can be found in [18]. These comparisons show that, especially for special element geometries, the performances are similar.

As a potential alternative or complement to the LSIC (least-squares on incompressibility constraint) stabilization, we propose the Discontinuity-Capturing Directional Dissipation (DCDD) stabilization. In introducing the DCDD stabilization, we first define the unit vectors $\mathbf{s}$ and $\mathbf{r}$:

$$\mathbf{s} = \frac{\mathbf{u}^h}{\|\mathbf{u}^h\|}, \tag{62}$$

$$\mathbf{r} = \frac{\boldsymbol{\nabla}\|\mathbf{u}^h\|}{\|\,\boldsymbol{\nabla}\|\mathbf{u}^h\|\,\|}, \tag{63}$$

and the element-level matrices and vectors $\mathbf{c}_r$, $\tilde{\mathbf{k}}_r$, $(\mathbf{c}_r)_{\mathrm{V}}$, and $(\tilde{\mathbf{k}}_r)_{\mathrm{V}}$:

$$\mathbf{c}_r : \quad \int_{\Omega^e} \mathbf{w}^h \cdot \rho(\mathbf{r}\cdot\boldsymbol{\nabla}\mathbf{u}^h) d\Omega \quad : (\mathbf{c}_r)_{\mathrm{V}} \ , \tag{64}$$

$$\tilde{\mathbf{k}}_r : \quad \int_{\Omega^e} (\mathbf{r}\cdot\boldsymbol{\nabla}\mathbf{w}^h) \cdot \rho(\mathbf{r}\cdot\boldsymbol{\nabla}\mathbf{u}^h) d\Omega \quad : (\tilde{\mathbf{k}}_r)_{\mathrm{V}} \ . \tag{65}$$

Then the DCDD stabilization is defined as

$$S_{\mathrm{DCDD}} = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \rho\nu_{\mathrm{DCDD}} \boldsymbol{\nabla}\mathbf{w}^h : \left(\left[\mathbf{rr} - (\mathbf{r}\cdot\mathbf{s})^2 \mathbf{ss} \right] \cdot \boldsymbol{\nabla}\mathbf{u}^h \right) d\Omega, \tag{66}$$

where the element-matrix-based and element-vector-based DCDD viscosities are defined as:

$$\nu_{\mathrm{DCDD}} = |\mathbf{r}\cdot\mathbf{u}^h| \frac{\|\mathbf{c}_r\|}{\|\tilde{\mathbf{k}}_r\|} \ , \tag{67}$$

$$(\nu_{\mathrm{DCDD}})_{\mathrm{V}} = |\mathbf{r}\cdot\mathbf{u}^h| \frac{\|(\mathbf{c}_r)_{\mathrm{V}}\|}{\|(\tilde{\mathbf{k}}_r)_{\mathrm{V}}\|} \ . \tag{68}$$

An approximate version of the expression given by Eq. (67) can be written as

$$\nu_{\mathrm{DCDD}} = |\mathbf{r}\cdot\mathbf{u}^h| \frac{h_{\mathrm{RGN}}}{2}, \tag{69}$$

where

$$h_{\mathrm{RGN}} = 2 \left(\sum_{a=1}^{n_{en}} |\mathbf{r}\cdot\boldsymbol{\nabla}N_a| \right)^{-1}. \tag{70}$$

A different way of determining ν_{DCDD} can be expressed as

$$\nu_{\mathrm{DCDD}} = \tau_{\mathrm{DCDD}} \|\mathbf{u}^h\|^2, \tag{71}$$

where

$$\tau_{\mathrm{DCDD}} = \frac{h_{\mathrm{DCDD}}}{2\|\mathbf{U}\|} \ \frac{\|\,\boldsymbol{\nabla}\|\mathbf{u}^h\|\,\|\, h_{\mathrm{DCDD}}}{\|\mathbf{U}\|} \ . \tag{72}$$

Here $\mathbf{U}$ represents a global velocity scale, and h_{DCDD} can be calculated by using the expression

$$h_{\text{DCDD}} = 2\ \frac{\|\mathbf{c}_r\|}{\|\tilde{\mathbf{k}_r}\|}\ , \tag{73}$$

or the approximation

$$h_{\text{DCDD}} = h_{\text{RGN}}\ . \tag{74}$$

Combining Eqs. (71) and (72), we obtain

$$\nu_{\text{DCDD}} = \frac{1}{2}\left(\frac{\|\mathbf{u}^h\|}{\|\mathbf{U}\|}\right)^2 (h_{\text{DCDD}})^2\ \|\ \boldsymbol{\nabla}\|\mathbf{u}^h\|\ \|\ . \tag{75}$$

4 CONCLUDING REMARKS

In this paper, we described adaptive ways of determining the stabilization parameters used in the stabilized finite element methods, particularly the streamline-upwind/Petrov-Galerkin (SUPG) and pressure-stabilizing/Petrov-Galerkin (PSPG) formulations. The parameters we proposed are calculated from the element-level matrices and vectors, without separately calculating local length scales. However, these parameters automatically take into account the local length scales, as well as the advection field and the element-level Reynolds number. We described these adaptive calculations of the stabilization parameters for a time-dependent advection diffusion equation and the Navier-Stokes equations of unsteady incompressible flows.

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Appendix I
INCLUSION OF THE FLUX JUMP TERMS IN THE STABILIZED FORMULATION

We start with the Galerkin formulation of Eq. (1):

$$\int_\Omega w^h \left(\frac{\partial \phi^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla}\phi^h\right) d\Omega + \int_\Omega \boldsymbol{\nabla} w^h \cdot \nu \boldsymbol{\nabla}\phi^h d\Omega - \int_{\Gamma_h} w^h h d\Gamma = 0. \tag{I.1}$$

Re-write the second term as follows:

$$\int_\Omega \boldsymbol{\nabla} w^h \cdot \nu \boldsymbol{\nabla}\phi^h d\Omega = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \boldsymbol{\nabla} w^h \cdot \nu \boldsymbol{\nabla}\phi^h d\Omega. \tag{I.2}$$

Then integrate-by-parts:

$$\int_{\Omega^e} \boldsymbol{\nabla} w^h \cdot \nu \boldsymbol{\nabla} \phi^h d\Omega = \int_{\Gamma^e} w^h \mathbf{n} \cdot \nu \boldsymbol{\nabla} \phi^h d\Gamma - \int_{\Omega^e} w^h \boldsymbol{\nabla} \cdot \left(\nu \boldsymbol{\nabla} \phi^h \right) d\Omega. \tag{I.3}$$

Furthermore:

$$\sum_{e=1}^{n_{el}} \int_{\Gamma^e} w^h \mathbf{n} \cdot \nu \boldsymbol{\nabla} \phi^h d\Gamma = \sum_{e=1}^{n_{el}} \int_{\Gamma^e_{if}} w^h \mathbf{n} \cdot \nu \boldsymbol{\nabla} \phi^h d\Gamma + \int_{\Gamma_h} w^h \mathbf{n} \cdot \nu \boldsymbol{\nabla} \phi^h d\Gamma, \tag{I.4}$$

where Γ^e_{if} is the interior boundary (interior faces) of the element e. The first term on the right-hand-side can also be written as a sum over interior faces:

$$\sum_{e=1}^{n_{el}} \int_{\Gamma^e_{if}} w^h \mathbf{n} \cdot \nu \boldsymbol{\nabla} \phi^h d\Gamma = \sum_{k=1}^{n_{if}} \int_{\Gamma_k} w^h \left(\mathbf{n}_1 \cdot \nu \boldsymbol{\nabla} \phi^h_1 + \mathbf{n}_2 \cdot \nu \boldsymbol{\nabla} \phi^h_2 \right) d\Gamma, \tag{I.5}$$

where n_{if} is the number of interior faces, Γ_k is the kth interior face, and the subscripts 1 and 2 refer to elements sharing that face (see Figure I.1).

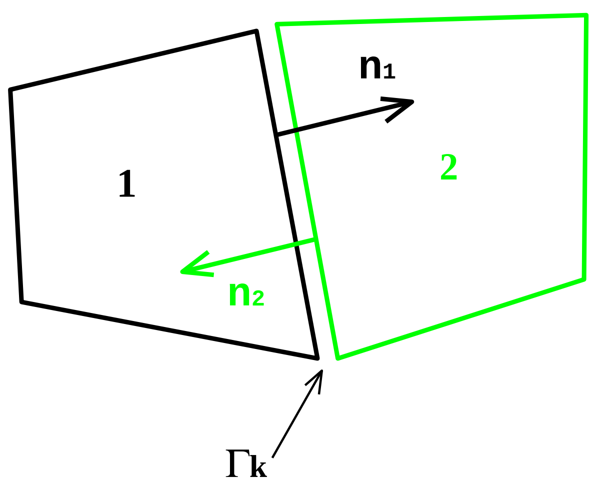

Figure I.1: The interior face.

Consequently, we can write:

$$\sum_{e=1}^{n_{el}} \int_{\Omega^e} w^h \left(\frac{\partial \phi^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \phi^h - \boldsymbol{\nabla} \cdot \left(\nu \boldsymbol{\nabla} \phi^h \right) \right) d\Omega + \sum_{e=1}^{n_{el}} \int_{\Gamma^e_{if}} w^h J d\Gamma + \int_{\Gamma_h} w^h \left(\mathbf{n} \cdot \nu \boldsymbol{\nabla} \phi^h - h \right) d\Gamma, \tag{I.6}$$

where $J = \mathbf{n}_1 \cdot \nu \boldsymbol{\nabla} \phi^h_1 + \mathbf{n}_2 \cdot \nu \boldsymbol{\nabla} \phi^h_2$ is the flux jump term.

Based on Eq. (I.6), we modify the stabilized formulation given by Eq. (5) as follows:

$$\int_{\Omega} w^h \left(\frac{\partial \phi^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \phi^h \right) d\Omega + \int_{\Omega} \boldsymbol{\nabla} w^h \cdot \nu \boldsymbol{\nabla} \phi^h d\Omega$$

$$
\begin{aligned}
&+ \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\text{SUPG}} \mathbf{u}^h \cdot \boldsymbol{\nabla} w^h \left(\frac{\partial \phi^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \phi^h - \boldsymbol{\nabla} \cdot \left(\nu \boldsymbol{\nabla} \phi^h \right) \right) d\Omega \\
&+ \sum_{e=1}^{n_{el}} \int_{\Gamma_{if}^e} \tau_{\text{SUPG}} \mathbf{u}^h \cdot \boldsymbol{\nabla} w^h \; J \; d\Gamma \\
&+ \int_{\Gamma_h} \tau_{\text{SUPG}} \mathbf{u}^h \cdot \boldsymbol{\nabla} w^h (\mathbf{n} \cdot \nu \boldsymbol{\nabla} \phi^h - h) d\Gamma \\
&= \int_{\Gamma_h} w^h h d\Gamma. \qquad \text{(I.7)}
\end{aligned}
$$

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