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NOTES ON

THE EQUIVALENCE OF SELECTIVE LUMPING AND NUMERICAL DIFFUSION APPROACHES

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$$m_{ab} = \int_{\Omega} N_a N_b \, d\Omega \qquad N_1 = \frac{1}{2}(1-\xi) \qquad N_2 = \frac{1}{2}(1+\xi)$$

$$m_{11} = \frac{h}{2}\frac{1}{4}\int_{-1}^{+1}(1-\xi)(1-\xi)\,d\xi = \frac{h}{8}\int_{-1}^{+1}\left(1-2\xi+\xi^2\right)d\xi = \frac{h}{8}\left(\xi+\frac{1}{3}\xi^3\right)\Big|_{-1}^{+1}$$

$$= \frac{h}{8}\underbrace{\left(2+\frac{2}{3}\right)}_{8/3} = \frac{h}{3}$$

$$m_{12} = \frac{h}{2}\frac{1}{4}\int_{-1}^{+1}(1-\xi)(1+\xi)\,d\xi = \frac{h}{8}\int_{-1}^{+1}\left(1 \quad -\xi^2\right)d\xi$$

$$= \frac{h}{8}\underbrace{\left(2-\frac{2}{3}\right)}_{4/3} = \frac{h}{6}$$

$$\underset{\sim}{m} = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \frac{h}{6}(1, 4, 1) \rightarrow \frac{h}{6}\ \bullet\overset{1}{\text{———}}\overset{4}{\bullet}\overset{1}{\text{———}}\bullet$$

$$\underset{\sim}{m}_L = \frac{h}{6}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \rightarrow \frac{h}{6}(0, 6, 0) \rightarrow \frac{h}{6}\ \bullet\overset{0}{\text{———}}\overset{6}{\bullet}\overset{0}{\text{———}}\bullet$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\underset{\sim}{m}_L - \underset{\sim}{m} = \frac{h}{6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \underset{\sim}{\frac{h}{6}}(-1, 2, -1) \rightarrow \frac{h}{6}\ \bullet\overset{-1}{\text{———}}\overset{2}{\bullet}\overset{-1}{\text{———}}\bullet$$

L

$$k_{ab} = \kappa \int_{\Omega} N_{a,x} N_{b,x}\, d\Omega = \kappa \frac{2}{h} \int_{-1}^{+1} N_{a,\xi} N_{b,\xi}\, d\xi \qquad N_{1,\xi} = -\frac{1}{2} \quad N_{2,\xi} = \frac{1}{2}$$

$$k_{11} = \kappa \frac{2}{h} \int_{-1}^{+1} \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) d\xi = \kappa \frac{2}{h} \cdot \frac{1}{2} = \kappa \frac{1}{h}$$

$$k_{12} = \qquad - \kappa \frac{1}{h}$$

$$\underset{\sim}{k} = \kappa \frac{1}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \underbrace{\overset{\frac{L^2}{T}}{\kappa} \frac{1}{h}}_{\frac{L}{T}} (-1,\ 2,\ -1) \rightarrow \kappa \frac{1}{h} \quad \underset{\bullet}{-1} \text{———} \underset{\bullet}{2} \text{———} \underset{\bullet}{-1}$$

—— o ——

$$c_{ab} = u \int_{\Omega} N_a N_{b,x}\, d\Omega = u \int_{-1}^{+1} N_a N_{b,\xi}\, d\xi$$

$$c_{11} = u \frac{1}{4} \int_{-1}^{+1} (1-\xi)(-1)\, d\xi = u \frac{1}{4} \left(-\xi\right)\Big|_{-1}^{+1} = -u \frac{1}{2}$$

$$c_{22} = \qquad = u \frac{1}{2}$$

$$c_{12} = \qquad = u \frac{1}{2}$$

$$c_{21} = \qquad = -u \frac{1}{2}$$

$$\underset{\sim}{c} = u \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \rightarrow \underbrace{u \frac{1}{2}}_{\frac{L}{T}} (-1,\ 0,\ 1) \rightarrow u \frac{1}{2} \quad \underset{\bullet}{-1} \text{———} \underset{\bullet}{0} \text{———} \underset{\bullet}{1}$$

$\boxed{6/29\text{-}3}$

"SUPG"

$$(w + \tau \underset{\sim}{u} \cdot \underset{\sim}{\nabla} w)(\phi_{,t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \phi)$$

OR

$$(w + C_{2\tau} \frac{h}{2} \underset{\sim}{s} \cdot \underset{\sim}{\nabla} w)(\quad " \quad), \quad \underset{\sim}{s} = \frac{\underset{\sim}{u}}{\|\underset{\sim}{u}\|}$$

$$\tau \underset{\sim}{u} = C_{2\tau} \frac{h}{2} \underset{\sim}{s}$$

$$\Rightarrow \quad \tau \|\underset{\sim}{u}\| \underset{\sim}{s} = C_{2\tau} \frac{h}{2} \underset{\sim}{s} \quad \Rightarrow \quad C_{2\tau} = \frac{(2\tau)\|\underset{\sim}{u}\|}{h}$$

$$\Rightarrow \quad \tau = C_{2\tau} \frac{h}{2\|\underset{\sim}{u}\|}$$

IF $C_{2\tau} = 1 \quad \Rightarrow \quad \tau = \frac{h}{2\|\underset{\sim}{u}\|}$

IF $C_{2\tau} = C_{\Delta t} = \frac{\Delta t \|\underset{\sim}{u}\|}{h} \quad \Rightarrow \quad \tau = \frac{\Delta t}{2}$

DIFFUSION TERM :

$$\underset{\sim}{\nabla} w \cdot \tau \underset{\sim}{u} \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \phi$$

$$\underset{\sim}{\nabla} w \cdot C_{2\tau} \frac{h}{2} \|\underset{\sim}{u}\| \underset{\sim}{s} \underset{\sim}{s} \cdot \underset{\sim}{\nabla} \phi$$

$$\underset{\sim}{\nabla} w \cdot \underset{\sim}{\tilde{\kappa}} \cdot \underset{\sim}{\nabla} \phi$$

$$\Rightarrow \quad \underset{\sim}{\tilde{\kappa}} = \tau \underset{\sim}{u} \underset{\sim}{u} = C_{2\tau} \frac{h}{2} \|\underset{\sim}{u}\| \underset{\sim}{s} \cdot \underset{\sim}{s}$$

1-D : $\quad \tilde{\kappa} = \tau |u|^2 = C_{2\tau} \frac{h}{2} |u|$

$\boxed{6/29-4}$

NUMERICAL DIFFUSION MATRIX

$$\underset{\sim}{\tilde{k}} = \tilde{\kappa}\,\frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \underbrace{\tilde{\kappa}\,\frac{1}{h}}_{\frac{L}{T}}\,(-1,\ 2,\ -1) \rightarrow \tilde{\kappa}\,\frac{1}{h}\quad -1 \quad 2 \quad -1$$

$$\frac{\underset{\sim}{m}_L - \underset{\sim}{m}}{\Delta t} = \frac{h}{6}\,\frac{1}{\Delta t}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \underbrace{\frac{h}{6}\,\frac{1}{\Delta t}}_{\frac{L}{T}}\,(-1,\ 2,\ -1)$$

FIND f SUCH THAT

$$f\,\frac{\underset{\sim}{m}_L - \underset{\sim}{m}}{\Delta t} = \underset{\sim}{\tilde{k}}$$

$$\Rightarrow \quad f\,\frac{h}{6}\,\frac{1}{\Delta t} = \tilde{\kappa}\,\frac{1}{h} \quad \Rightarrow f = 6\,\frac{\Delta t}{h^2}\,\tilde{\kappa}$$

BUT $\tilde{\kappa} = \tau\,|u|^2 = C_{2\tau}\,\frac{h}{2}\,|u|$

$$\Rightarrow \quad f = C_{2\tau}\ 3\ \frac{\Delta t\,|u|}{h} = C_{2\tau}\ 3\ C_{\Delta t}$$

$\Rightarrow$ IF $C_{2\tau} = 1 \quad \Rightarrow \quad f = 3\,C_{\Delta t}$

IF $C_{2\tau} = C_{\Delta t} \quad \Rightarrow \quad f = 3\,(C_{\Delta t})^2$

$\boxed{6/29\_5}$

SELECTIVE LUMPING

$$\underset{\sim}{M}_L \underset{\sim}{\phi}_{n+\alpha} - \tilde{\underset{\sim}{M}} \underset{\sim}{\phi}_n \quad , \quad \alpha = \begin{cases} 1/2 \\ 1 \end{cases}$$

$$\begin{aligned} \tilde{\underset{\sim}{M}} &= e\underset{\sim}{M}_L + (1-e)\underset{\sim}{M} \\ &= e\underset{\sim}{M}_L + (1-e)(\underset{\sim}{M} - \underset{\sim}{M}_L + \underset{\sim}{M}_L) \\ &= \underset{\sim}{M}_L - (1-e)(\underset{\sim}{M}_L - \underset{\sim}{M}) \end{aligned}$$

$$\Rightarrow \quad \frac{1}{\Delta t}\left(\underset{\sim}{M}_L \underset{\sim}{\phi}_{n+\alpha} - \tilde{\underset{\sim}{M}} \underset{\sim}{\phi}_n\right)$$

$$= \underset{\sim}{M}_L \frac{(\underset{\sim}{\phi}_{n+\alpha} - \underset{\sim}{\phi}_n)}{\Delta t} + \underbrace{(1-e)\frac{(\underset{\sim}{M}_L - \underset{\sim}{M})}{\Delta t}\underset{\sim}{\phi}_n}_{\text{NUMERICAL DIFFUSION}}$$

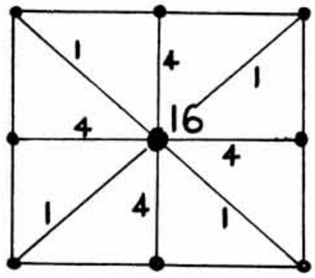
$$1-e = f \quad \Rightarrow \quad e = 1-f = 1 - 3C_{22}C_{\Delta t}$$

NOTE THAT $e < 0$ FOR MOST OF THE PRACTICAL SITUATIONS.

$\boxed{6/30 - \mathrm{I}}$

2 - DIMENSIONS

$$\underset{\sim}{m} = \frac{h^2}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} \rightarrow \quad \frac{h^2}{36}$$

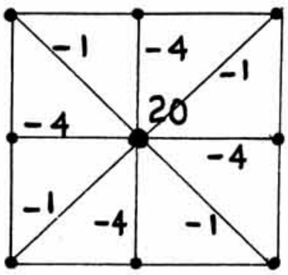


$$\rightarrow \frac{h^2}{36}\left(1, 4, 1, 4, 16, 4, 1, 4, 1\right)$$

$$\underset{\sim}{m}_L = \frac{h^2}{36}\begin{bmatrix} 9 & & & \\ & 9 & & \\ & & 9 & \\ & & & 9 \end{bmatrix} \rightarrow \frac{h^2}{36}\left(0, 0, 0, 0, 36, 0, 0, 0, 0\right)$$

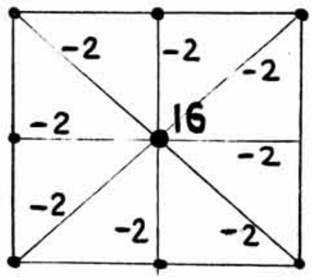
$$\underset{\sim}{m}_L - \underset{\sim}{m} = \frac{h^2}{36}\begin{bmatrix} 5 & -2 & -1 & -2 \\ -2 & 5 & -2 & -1 \\ -1 & -2 & 5 & -2 \\ -2 & -1 & -2 & 5 \end{bmatrix} \rightarrow \frac{h^2}{36}\left(-1, -4, -1, -4, 20, -4, -1, -4, -1\right)$$

$$\rightarrow \quad \frac{h^2}{36}$$



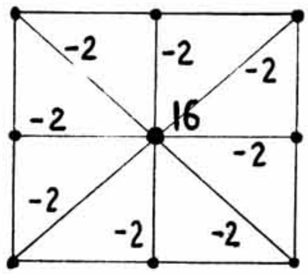
L^2

$$\underset{\sim}{k} = \frac{\kappa}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix} \rightarrow \underbrace{\frac{\kappa}{6}}_{\frac{L^2}{T}}$$



(ISOTROPIC)
NUMERICAL DIFFUSION COEFFICIENT

$$\underset{\sim}{\tilde{k}} = \qquad \rightarrow \underbrace{\frac{\tilde{\kappa}}{6}}_{\frac{L^2}{T}}$$



$$\underset{\sim}{\tilde{k}} = c_{2\tau}\frac{h}{2}\,\|\underset{\sim}{u}\|\,(\underset{\sim}{s}\cdot\underset{\sim}{s})$$

$\sim \underset{\sim}{I}$ (ASSUME ISOTROPIC DIFFUSION)

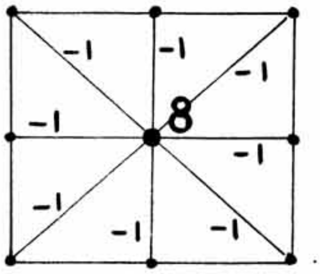
$$\Rightarrow \quad \tilde{\kappa} = c_{2\tau}\frac{h}{2}\,\|\underset{\sim}{u}\|$$

$$= \tau\,\|\underset{\sim}{u}\|^2$$

$\boxed{7/1-2}$

REWRITE $\underset{\sim}{m}_L - \underset{\sim}{m}$ AND $\tilde{\underset{\sim}{k}}$

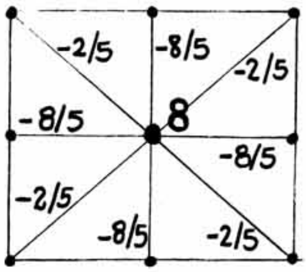
$\tilde{\underset{\sim}{k}} = \rightarrow \dfrac{\tilde{\kappa}}{3}$



$= \dfrac{\tilde{\kappa}}{3}\left(\underbrace{1\times}\ + 2\times\ \right)$

FINITE DIFFERENCE

$\dfrac{\underset{\sim}{m}_L - \underset{\sim}{m}}{\Delta t} = \rightarrow \dfrac{5}{72}\,\dfrac{h^2}{\Delta t}$



$\boxed{7/1-3}$

REWRITE $\underset{\sim}{\tilde{k}}$ AND $\underset{\sim}{m}_L - \underset{\sim}{m}$

$$\underset{\sim}{\tilde{k}} = \tilde{\kappa}\,\frac{1}{3}\left[\underbrace{\begin{pmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix}}_{\substack{\text{(FINITE DIFFERENCES)}^{*} \\ \text{HORIZONTAL-VERTICAL}}} + \underbrace{\begin{pmatrix} -1 & & -1 \\ & 4 & \\ -1 & & -1 \end{pmatrix}}_{\substack{2\times\text{(FINITE DIFFERENCES)}^{*} \\ \text{AT } 45^\circ}}\right]$$

$\equiv$  $\equiv 2 \times$ 

$\equiv \boxplus \qquad \equiv 2\,\boxtimes$

$*$ (I MEAN 5-POINT STENCIL)

$$\Rightarrow \underset{\sim}{\tilde{k}} = \tilde{\kappa}\,\frac{1}{3}\left(\boxplus + 2\boxtimes\right)$$

$$\frac{\underset{\sim}{m}_L - \underset{\sim}{m}}{\Delta t} = \frac{1}{6}\,\frac{h^2}{\Delta t}\,\frac{1}{12}\begin{pmatrix} -2 & -8 & -2 \\ -8 & 40 & -8 \\ -2 & -8 & -2 \end{pmatrix}$$

$$= \frac{1}{6}\,\frac{h^2}{\Delta t}\,\frac{1}{12}\left[\begin{pmatrix} & -8 & \\ -8 & 32 & -8 \\ & -8 & \end{pmatrix} + \begin{pmatrix} -2 & & -2 \\ & 8 & \\ -2 & & -2 \end{pmatrix}\right]$$

$$= \frac{1}{6}\,\frac{h^2}{\Delta t}\,\frac{1}{12}\left[8\begin{pmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix} + 2\begin{pmatrix} -1 & & -1 \\ & 4 & \\ -1 & & -1 \end{pmatrix}\right]$$

$\boxed{7/1\_4}$

$$= \frac{1}{6}\,\frac{h^2}{\Delta t}\,\frac{1}{12}\left(8\,\boxplus + 4\,\boxtimes\right)$$

$$= \frac{1}{6}\,\frac{h^2}{\Delta t}\,\frac{1}{3}\left(2\,\boxplus + \boxtimes\right)$$

SUMMARY :

$$\underset{\sim}{\tilde{k}} = \tilde{\kappa} \quad \frac{1}{3}\left(\boxplus + 2\,\boxtimes\right)$$

$$f\,\frac{\underset{\sim}{m}_L - \underset{\sim}{m}}{\Delta t} = f\,\frac{1}{6}\,\frac{h^2}{\Delta t} \quad \frac{1}{3}\left(2\,\boxplus + \boxtimes\right)$$

COMPARING THE TWO, AND ASSUMING THAT

$$\frac{1}{3}\left(\boxplus + 2\,\boxtimes\right) \cong \frac{1}{3}\left(2\,\boxplus + \boxtimes\right)$$

WE GET $\quad f\,\frac{1}{6}\,\frac{h^2}{\Delta t} = \tilde{\kappa} \quad \Rightarrow \quad f = 6\,\frac{\Delta t}{h^2}\,\tilde{\kappa}$

SINCE WE ALSO ASSUMED THAT $\quad \tilde{\kappa} = C_{2\tau}\,\frac{h}{2}\,\|\underset{\sim}{u}\| = \tau\,\|\underset{\sim}{u}\|^2$

$$\Rightarrow \quad f = C_{2\tau}\;3\;\frac{\Delta t\,\|\underset{\sim}{u}\|}{h} = C_{2\tau}\,3\,C_{\Delta t}$$; THE SAME AS BEFORE

$\boxed{7/2\_1}$

$$\underset{\sim}{M}\,\underset{\sim}{a} + \underset{\sim}{C}\,\underset{\sim}{v} - \underset{\sim}{F} = \underset{\sim}{0}$$

$$\frac{\underset{\sim}{M}}{\Delta t}\left(\underset{\sim}{v}_{n+1} - \underset{\sim}{v}_n\right) + \underset{\sim}{C}\left((1-\alpha)\underset{\sim}{v}_n + \alpha\,\underset{\sim}{v}_{n+1}\right) - \underset{\sim}{F}_{n+\alpha} = \underset{\sim}{0}$$

$$\left(\frac{\underset{\sim}{M}}{\Delta t} + \alpha\,\underset{\sim}{C}\right)\Delta\underset{\sim}{v}^{\,i}_{n+1} + \frac{\underset{\sim}{M}}{\Delta t}\left(\underset{\sim}{v}^{\,i}_{n+1} - \underset{\sim}{v}_n\right) + \underset{\sim}{C}\left((1-\alpha)\underset{\sim}{v}_n + \alpha\,\underset{\sim}{v}^{\,i}_{n+1}\right) - \underset{\sim}{F}_{n+\alpha} = \underset{\sim}{0}$$

$$\frac{\underset{\sim}{M}}{\Delta t}\left(\underset{\sim}{v}^{\,i+1}_{n+1} - \underset{\sim}{v}_n\right) + \alpha\,\underset{\sim}{C}\,\Delta\underset{\sim}{v}^{\,i}_{n+1} + \underset{\sim}{C}\left((1-\alpha)\underset{\sim}{v}_n + \alpha\,\underset{\sim}{v}^{\,i}_{n+1}\right) - \underset{\sim}{F}_{n+\alpha} = \underset{\sim}{0}$$

EXPLICIT SCHEMES

(EC) $$\frac{\underset{\sim}{M}_L}{\Delta t}\Delta\underset{\sim}{v}^{\,i}_{n+1} + \underbrace{\frac{\underset{\sim}{M}}{\Delta t}\left(\underset{\sim}{v}^{\,i}_{n+1} - \underset{\sim}{v}_n\right) + \underset{\sim}{C}\left((1-\alpha)\underset{\sim}{v}_n + \alpha\,\underset{\sim}{v}^{\,i}_{n+1}\right) - \underset{\sim}{F}_{n+\alpha}}_{\ldots\ \text{CONSISTENT}\ \ldots} = \underset{\sim}{0}$$

(EL) $$\frac{\underset{\sim}{M}_L}{\Delta t}\Delta\underset{\sim}{v}^{\,i}_{n+1} + \underbrace{\frac{\underset{\sim}{M}_L}{\Delta t}\left(\underset{\sim}{v}^{\,i}_{n+1} - \underset{\sim}{v}_n\right) + \underset{\sim}{C}\left((1-\alpha)\underset{\sim}{v}_n + \alpha\,\underset{\sim}{v}^{\,i}_{n+1}\right) - \underset{\sim}{F}_{n+\alpha}}_{\ldots\ \text{LUMPED}\ \ldots} = \underset{\sim}{0}$$

(ES) $$\frac{\underset{\sim}{M}_L}{\Delta t}\Delta\underset{\sim}{v}^{\,i}_{n+1} + \underbrace{\frac{\underset{\sim}{M}_L}{\Delta t}\underset{\sim}{v}^{\,i}_{n+1} - \frac{\tilde{\underset{\sim}{M}}}{\Delta t}\underset{\sim}{v}_n + \underset{\sim}{C}\left((1-\alpha)\underset{\sim}{v}_n + \alpha\,\underset{\sim}{v}^{\,i}_{n+1}\right) - \underset{\sim}{F}_{n+\alpha}}_{\ldots\ \text{SELECTIVELY LUMPED}\ \ldots} = \underset{\sim}{0}$$

$$\frac{\tilde{\underset{\sim}{M}}}{\Delta t} \equiv \left((1-f)\,\underset{\sim}{M}_L + f\,\underset{\sim}{M}\right)/\Delta t$$

BUT $$f\,\frac{\underset{\sim}{M}_L - \underset{\sim}{M}}{\Delta t} = \tilde{\underset{\sim}{K}} \;\Rightarrow\; f\,\frac{\underset{\sim}{M}}{\Delta t} = f\,\frac{\underset{\sim}{M}_L}{\Delta t} - \tilde{\underset{\sim}{K}}$$

$$\Rightarrow\quad \frac{\tilde{\underset{\sim}{M}}}{\Delta t} = (1-f)\frac{\underset{\sim}{M}_L}{\Delta t} + f\,\frac{\underset{\sim}{M}_L}{\Delta t} - \tilde{\underset{\sim}{K}} = \frac{\underset{\sim}{M}_L}{\Delta t} - \tilde{\underset{\sim}{K}}$$

$\therefore$

$$(ES)\quad \frac{\mathbf{M}_L}{\Delta t}\,\Delta\mathbf{V}^{i}_{n+1} + \overbrace{\frac{\mathbf{M}_L}{\Delta t}\left(\mathbf{V}^{i}_{n+1}-\mathbf{V}_n\right)}^{\ldots\text{LUMPED}\ldots} + \underbrace{\tilde{\mathbf{K}}\,\mathbf{V}_n}_{\text{NUMERICAL DIFFUSION}} + \overbrace{\mathbf{C}\left((1-\alpha)\mathbf{V}_n+\alpha\mathbf{V}^{i}_{n+1}\right)-\mathbf{F}_{n+\alpha}}^{\ldots\text{LUMPED}\ldots} = 0$$

$\sim\circ\sim$

$$(EL)\quad \frac{\mathbf{M}_L}{\Delta t}\left(\mathbf{V}^{i+1}_{n+1}-\mathbf{V}_n\right) + \mathbf{C}\left((1-\alpha)\mathbf{V}_n+\alpha\mathbf{V}^{i}_{n+1}\right)-\mathbf{F}_{n+\alpha} = \mathbf{0}$$

$$(ES)\quad \frac{\mathbf{M}_L}{\Delta t}\left(\mathbf{V}^{i+1}_{n+1}-\mathbf{V}_n\right) + \overbrace{\tilde{\mathbf{K}}\,\mathbf{V}_n}^{\text{NUMERICAL DIFFUSION}} + \mathbf{C}\left((1-\alpha)\mathbf{V}_n+\alpha\mathbf{V}^{i}_{n+1}\right)-\mathbf{F}_{n+\alpha} = \mathbf{0}$$

$$(EC)\quad \frac{\mathbf{M}_L}{\Delta t}\left(\mathbf{V}^{i+1}_{n+1}-\mathbf{V}^{i}_{n+1}\right)+\frac{\mathbf{M}}{\Delta t}\left(\mathbf{V}^{i}_{n+1}-\mathbf{V}_n\right) + \mathbf{C}\left((1-\alpha)\mathbf{V}_n+\alpha\,\mathbf{V}^{i}_{n+1}\right)-\mathbf{F}_{n+\alpha} = \mathbf{0}$$

$\boxed{7/2\text{-}3}$

(EC) $\alpha = 1/2$

$$\mathbf{V}^{i+1}_{n+1} = \mathbf{V}^{i}_{n+1} - \underbrace{\mathbf{M}_L^{-1}\mathbf{M}}_{\mathbf{J}}\left(\mathbf{V}^{i}_{n+1} - \mathbf{V}_n\right) - \underbrace{\frac{\Delta t}{2}\mathbf{M}_L^{-1}\mathbf{C}}_{\mathbf{D}}\left(\mathbf{V}^{i}_{n+1} + \mathbf{V}_n\right)$$

$$\mathbf{V}^{i+1}_{n+1} = \underbrace{(\mathbf{J} - \mathbf{D})}_{\mathbf{A}_1}\mathbf{V}_n + \underbrace{(\mathbf{I} - \mathbf{J} - \mathbf{D})}_{\mathbf{A}_2}\mathbf{V}^{i}_{n+1}$$

$$\mathbf{V}^{1}_{n+1} = \mathbf{A}_1\mathbf{V}_n + \mathbf{A}_2\mathbf{V}^{0}_{n+1}$$

$$\mathbf{V}^{2}_{n+1} = \mathbf{A}_1\mathbf{V}_n + \mathbf{A}_2\mathbf{V}^{1}_{n+1} = \mathbf{A}_1\mathbf{V}_n + \mathbf{A}_2\left(\mathbf{A}_1\mathbf{V}_n + \mathbf{A}_2\mathbf{V}^{0}_{n+1}\right)$$

$$= \left(\mathbf{A}_1 + \mathbf{A}_2\mathbf{A}_1\right)\mathbf{V}_n + \mathbf{A}_2^2\mathbf{V}^{0}_{n+1}$$

$$= \mathbf{A}_1\mathbf{V}_n + \mathbf{A}_2\left(\mathbf{A}_2 + \mathbf{A}_1 - \mathbf{A}_2\right)\mathbf{V}_n + \mathbf{A}_2^2\mathbf{V}^{0}_{n+1}$$

$$= \mathbf{A}_1\mathbf{V}_n + \mathbf{A}_2\left(\mathbf{A}_1 - \mathbf{A}_2\right)\mathbf{V}_n + 2\mathbf{A}_2^2\left(\frac{\mathbf{V}_n + \mathbf{V}^{0}_{n+1}}{2}\right)$$

$$\begin{aligned}
\mathbf{A}_2(\mathbf{A}_1 - \mathbf{A}_2) &= (\mathbf{I} - \mathbf{J} - \mathbf{D})(\mathbf{J} - \mathbf{D} - \mathbf{I} + \mathbf{J} + \mathbf{D}) \\
&= (\mathbf{I} - \mathbf{J} - \mathbf{D})(2\mathbf{J} - \mathbf{I}) \\
&= 3\mathbf{J} - \mathbf{I} - 2\mathbf{J}^2 - 2\mathbf{D}\mathbf{J} + \mathbf{D} \\
\mathbf{A}_1 &= \mathbf{J} \qquad\qquad\qquad\qquad\qquad - \mathbf{D}
\end{aligned}$$

$$\Rightarrow \mathbf{V}^{2}_{n+1} = \left(4\mathbf{J} - \mathbf{I} - 2\mathbf{J}^2\right)\mathbf{V}_n - \left(2\mathbf{D}\mathbf{J}\right)\mathbf{V}_n + 2\mathbf{A}_2^2\left(\frac{\mathbf{V}_n + \mathbf{V}^{0}_{n+1}}{2}\right)$$

IF $\mathbf{V}^{0}_{n+1} = \mathbf{V}_n$

$$\Rightarrow \mathbf{V}^{2}_{n+1} = \left[\mathbf{I} - (2\mathbf{I} - \mathbf{J})2\mathbf{D} + 2\mathbf{D}^2\right]\mathbf{V}_n$$

$$= \left[\mathbf{I} - 2\mathbf{D} + 2\mathbf{D}^2\right]\mathbf{V}_n - (\mathbf{I} - \mathbf{J})2\mathbf{D}\mathbf{V}_n$$

$\boxed{7/2-4}$

(EL) $\alpha = 1/2$

$$\underset{\sim}{M} = \underset{\sim}{M}_L \quad \Rightarrow \quad \underset{\sim}{J} = \underset{\sim}{M}_L^{-1} \underset{\sim}{M} = \underset{\sim}{I}$$

$$\Rightarrow \quad \underset{\sim}{V}_{n+1}^{2} = \underset{\sim}{V}_n - 2\underset{\sim}{D}\,\underset{\sim}{V}_n + 2\underset{\sim}{D}^2 \frac{\left(\underset{\sim}{V}_n + \underset{\sim}{V}_{n+1}^{0}\right)}{2}$$

$$\text{IF} \quad \underset{\sim}{V}_{n+1}^{0} = \underset{\sim}{V}_n$$

$$\Rightarrow \quad \underset{\sim}{V}_{n+1}^{2} = \underset{\sim}{V}_n - 2\underset{\sim}{D}\,\underset{\sim}{V}_n + 2\underset{\sim}{D}^2 \underset{\sim}{V}_n$$

(2-STEP METHOD)

$$\underset{\sim}{V}_{n+1/2} = \underset{\sim}{V}_n - \underbrace{\frac{\Delta t}{2} \underset{\sim}{M}_L^{-1} \underset{\sim}{C}}_{\underset{\sim}{D}}\, \underset{\sim}{V}_n = (\underset{\sim}{I} - \underset{\sim}{D})\, \underset{\sim}{V}_n$$

$$\underset{\sim}{V}_{n+1} = \underset{\sim}{V}_n - \overbrace{\Delta t\, \underset{\sim}{M}_L^{-1} \underset{\sim}{C}}^{2\underset{\sim}{D}}\, \underset{\sim}{V}_{n+1/2} = \underset{\sim}{V}_n - 2\underset{\sim}{D}\, \underset{\sim}{V}_{n+1/2}$$

$$= \underset{\sim}{V}_n - 2\underset{\sim}{D}\,(\underset{\sim}{I} - \underset{\sim}{D})\, \underset{\sim}{V}_n = \underset{\sim}{V}_n - 2\underset{\sim}{D}\,\underset{\sim}{V}_n + 2\underset{\sim}{D}^2 \underset{\sim}{V}_n$$

SELECTIVE LUMPING

1-STEP

$$\underset{\sim}{V}_{n+1} = \underset{\sim}{V}_n - \underbrace{\Delta t\, \underset{\sim}{M}_L^{-1} \tilde{\underset{\sim}{K}}}_{2\tilde{\underset{\sim}{L}}} \underset{\sim}{V}_n - \underbrace{\Delta t\, \underset{\sim}{M}_L^{-1} \underset{\sim}{C}}_{2\underset{\sim}{D}} \underset{\sim}{V}_n \qquad f = 3 C_{2\tau} C_{\Delta t}$$

$$\Rightarrow \underset{\sim}{V}_{n+1} = \underset{\sim}{V}_n - 2\underset{\sim}{D}\,\underset{\sim}{V}_n - 2\tilde{\underset{\sim}{L}}\,\underset{\sim}{V}_n$$

2-STEP

$$\underset{\sim}{V}_{n+1/2} = \underset{\sim}{V}_n - \frac{\Delta t}{2} \underset{\sim}{M}_L^{-1} \underset{\sim}{C}\, \underset{\sim}{V}_n - \frac{\Delta t}{2} \underset{\sim}{M}_L^{-1} \tilde{\underset{\sim}{K}}\, \underset{\sim}{V}_n \quad , \quad f = \frac{3}{2} C_{2\tau} C_{\Delta t}$$

$$= \underset{\sim}{V}_n - \underset{\sim}{D}\,\underset{\sim}{V}_n - \tilde{\underset{\sim}{L}}\,\underset{\sim}{V}_n \; =$$

$$\underset{\sim}{V}_{n+1} = \underset{\sim}{V}_n - \Delta t\, \underset{\sim}{M}_L^{-1} \underset{\sim}{C}\, \underset{\sim}{V}_{n+1/2} - \Delta t\, \underset{\sim}{M}_L^{-1} \tilde{\underset{\sim}{K}}\, \underset{\sim}{V}_n \quad , \quad f = 3 C_{2\tau} C_{\Delta t}$$

$$= \underset{\sim}{V}_n - 2\underset{\sim}{D}\,\underset{\sim}{V}_{n+1/2} - 2\tilde{\underset{\sim}{L}}\,\underset{\sim}{V}_n$$

$$= \underset{\sim}{V}_n - 2\underset{\sim}{D}\left(\underset{\sim}{V}_n - \underset{\sim}{D}\,\underset{\sim}{V}_n - \tilde{\underset{\sim}{L}}\,\underset{\sim}{V}_n\right) - 2\tilde{\underset{\sim}{L}}\,\underset{\sim}{V}_n$$

$$= \underset{\sim}{V}_n - 2\underset{\sim}{D}\,\underset{\sim}{V}_n + 2\underset{\sim}{D}^2\underset{\sim}{V}_n + 2\underset{\sim}{D}\tilde{\underset{\sim}{L}}\,\underset{\sim}{V}_n - 2\tilde{\underset{\sim}{L}}\,\underset{\sim}{V}_n$$

$$= \underset{\sim}{V}_n - 2\underset{\sim}{D}\,\underset{\sim}{V}_n + 2\underset{\sim}{D}^2\underset{\sim}{V}_n - 2\tilde{\underset{\sim}{L}}\,\underset{\sim}{V}_n + 2\underset{\sim}{D}\tilde{\underset{\sim}{L}}\,\underset{\sim}{V}_n$$

$$= \underset{\sim}{V}_n - 2(\underset{\sim}{D} + \tilde{\underset{\sim}{L}})\,\underset{\sim}{V}_n + 2\underset{\sim}{D}\,(\underset{\sim}{D} + \tilde{\underset{\sim}{L}})\,\underset{\sim}{V}_n$$