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# DETERMINATION OF THE STABILIZATION AND SHOCK-CAPTURING PARAMETERS IN SUPG FORMULATION OF COMPRESSIBLE FLOWS

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**Abstract.** *The Streamline-Upwind/Petrov-Galerkin (SUPG) formulation is one of the most widely used stabilized methods in finite element computation of compressible flows. The formulation includes a stabilization parameter, which is mostly known as "$\tau$" and plays a significant part in determining the accuracy of the solution. Typically the SUPG formulation is used in combination with a shock-capturing term that provides additional stability near the shock fronts. The definition of the shock-capturing term includes a shock-capturing parameter that plays an important role in determining the quality of the solution near the shocks. In this paper, we describe, for the finite element formulation of compressible flows based on conservation variables, new ways for determining the $\tau$ and the shock-capturing parameter. The new definitions for the shock-capturing parameter are far simpler than the one based on the entropy variables and involve less operations in calculating the shock-capturing term.*

## 1 INTRODUCTION

Most of the computational fluid mechanics techniques reported in the literature in the past two decades are based on stabilized formulations. In finite element computation of flow problems, the Streamline-Upwind/Petrov-Galerkin (SUPG) formulation for incompressible flows [1, 2], the SUPG formulation for compressible flows [3, 4, 5], and the Pressure-Stabilizing/Petrov-Galerkin (PSPG) formulation for incompressible flows [6] are some of the most prevalent stabilized methods. Stabilized formulations such as the SUPG and PSPG formulations have a number of well-known advantages. They prevent numerical instabilities in solving problems with high Reynolds or Mach numbers and shocks or thin boundary layers, as well as when using equal-order interpolation functions for velocity and pressure. They also substantially improve the convergence rate in iterative solution of the large, coupled nonlinear equation system that needs to be solved at every time step of a flow computation. The SUPG and PSPG formulations are among the stabilized methods that achieve these objectives without introducing excessive numerical dissipation.

The SUPG formulation for incompressible flows was first introduced in an ASME paper [1], with further studies and examples in [2]. The SUPG formulation for compressible flows was first introduced, in the context of conservation variables, in a NASA technical report [3]. A concise version of the technical report was published as an AIAA paper [4], and a more thorough version with additional examples as a journal paper [5]. After that, several SUPG-like methods for compressible flows were developed. Taylor–Galerkin method [7], for example, is very similar, and under certain conditions is identical, to one of the SUPG methods introduced in [3, 4, 5]. Another example of the subsequent SUPG-like methods for compressible flows in conservation variables is the streamline-diffusion method described in [8]. Later, following [3, 4, 5], the SUPG formulation for compressible flows was recast in entropy variables and supplemented with a shock-capturing term [9]. It was shown in [10] that the SUPG formulation introduced in [3, 4, 5], when supplemented with a similar shock-capturing term, is very comparable in accuracy to the one that was recast in entropy variables. The stabilized formulation introduced in [11] for advection–diffusion–reaction equations also included a shock-capturing (discontinuity-capturing) term, and accounted for the interaction between the discontinuity-capturing and SUPG terms. The formulation precluded augmentation of the SUPG effect by the discontinuity-capturing effect when the advection and discontinuity directions coincide. The PSPG formulation for the Navier–Stokes equations of incompressible flows, introduced in [6], assures numerical stability while allowing us to use equal-order interpolation functions for velocity and pressure. An earlier version of this stabilized formulation for Stokes flow was introduced in [12].

A stabilization parameter, which is almost always known as "$\tau$", is embedded in the SUPG and PSPG formulations. It involves a measure of the local length scale (also known as "element length") and other parameters such as the element Reynolds and

Courant numbers. Judicious selection of the stabilization parameter plays an important role in determining the accuracy of the formulation. Various element lengths and $\tau$s were proposed starting with those in [1, 2] and [3, 4, 5], followed by the one introduced in [11], and those proposed in the subsequently reported SUPG and PSPG methods. Here we will call the SUPG formulation introduced in [3, 4, 5] for compressible flows "$(SUPG)_{82}$", and the set of $\tau$s introduced in conjunction with that formulation "$\tau_{82}$". The $\tau$ used in [10] with $(SUPG)_{82}$ is a slightly modified version of $\tau_{82}$. A shock-capturing parameter, which we will call here "$\delta_{91}$", was embedded in the shock-capturing term used in [10]. Subsequent minor modifications of $\tau_{82}$ took into account the interaction between the shock-capturing and the $(SUPG)_{82}$ terms in a fashion similar to how it was done in [11] for advection–diffusion–reaction equations. All these slightly modified versions of $\tau_{82}$ have always been used with the same $\delta_{91}$, and we will categorize them here all under the label "$\tau_{82\text{-MOD}}$".

To be used in conjunction with the SUPG/PSPG formulation of incompressible flows, the Discontinuity-Capturing Directional Dissipation (DCDD) stabilization was introduced in [13, 14, 15] for computation of flow fields with sharp gradients. This involved a second element length scale, which was also also introduced in [13, 14, 15] and is based on the solution gradient. This new element length scale is used together with the element length scales already defined in [11]. Recognizing this second element length as a diffusion length scale, new stabilization parameters for the diffusive limit were introduced in [14, 15, 16, 17, 18, 19]. The DCDD stabilization was originally conceived in [13, 14, 15] as an alternative to the LSIC (least-squares on incompressibility constraint) stabilization. The DCDD stabilization takes effect where there is a sharp gradient in the velocity field and introduces dissipation in the direction of that gradient. The way the DCDD stabilization is added to the finite element formulation precludes augmentation of the SUPG effect by the DCDD effect when the advection and discontinuity directions coincide.

Partly based on the ideas underlying the new $\tau$s for incompressible flows and the DCDD stabilization, new ways of calculating the $\tau$s and shock-capturing parameters for compressible flows were proposed in [18, 19, 20, 21, 22]. In this paper, we describe how the new parameters are defined. Like the $\tau$s and shock-capturing parameters developed earlier, these new parameters are intended for use with the SUPG formulation of compressible flows based on conservation variables. Compared to the earlier one that was derived based on the entropy variables, the new shock-capturing parameter is much simpler and computationally less costly.

The Navier–Stokes equations are given in Section 2. We summarize the stabilized formulations in Section 3, and describe the calculation of the stabilization parameters for incompressible flows in Section 4. The DCDD stabilization is described in Section 5. In Section 6, we describe the calculation of the stabilization parameters for compressible flows and the shock-capturing term. The concluding remarks are given in Section 7.

## 2 NAVIER–STOKES EQUATIONS

Let $\Omega \subset I\!R^{n_{sd}}$ be the spatial domain with boundary $\Gamma$, and $(0,T)$ be the time domain. The symbols $\rho$, $\mathbf{u}$, $p$ and $e$ will represent the density, velocity, pressure and the total energy, respectively. The external forces (e.g., the gravity) are represented by $\mathbf{f}$.

### 2.1 Compressible flows

The Navier–Stokes equations of compressible flows can be written on $\Omega$ and $\forall t \in (0,T)$ as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} - \frac{\partial \mathbf{E}_i}{\partial x_i} - \mathbf{R} = \mathbf{0} \ , \tag{1}$$

where $\mathbf{U} = (\rho, \rho u_1, \rho u_2, \rho u_3, \rho e)$ is the vector of conservation variables, and $\mathbf{F}_i$ and $\mathbf{E}_i$ are, respectively, the Euler and viscous flux vectors, defined as

$$\mathbf{F}_i = \begin{pmatrix} u_i \rho \\ u_i \rho u_1 + \delta_{i1} p \\ u_i \rho u_2 + \delta_{i2} p \\ u_i \rho u_3 + \delta_{i3} p \\ u_i(\rho e + p) \end{pmatrix} , \tag{2}$$

$$\mathbf{E}_i = \begin{pmatrix} 0 \\ T_{i1} \\ T_{i2} \\ T_{i3} \\ -q_i + T_{ik} u_k \end{pmatrix} . \tag{3}$$

Here $\delta_{ij}$ are the components of the identity tensor, $q_i$ are the components of the heat flux vector, and $T_{ij}$ are the components of the Newtonian viscous stress tensor:

$$\mathbf{T} = 2\mu \boldsymbol{\varepsilon}(\mathbf{u}), \tag{4}$$

where $\mu = \rho\nu$ is the viscosity, $\nu$ is the kinematic viscosity, and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the strain-rate tensor:

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left((\boldsymbol{\nabla}\mathbf{u}) + (\boldsymbol{\nabla}\mathbf{u})^T\right) . \tag{5}$$

The equation of state used here corresponds to the ideal gas assumption. The term $\mathbf{R}$ represents all other components that might enter the equations, including the external forces.

Eq. (1) can further be written in the following form:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}_i \frac{\partial \mathbf{U}}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\mathbf{K}_{ij} \frac{\partial \mathbf{U}}{\partial x_j}\right) - \mathbf{R} = \mathbf{0} \ , \tag{6}$$

where

$$\mathbf{A}_i = \frac{\partial \mathbf{F}_i}{\partial \mathbf{U}}, \tag{7}$$

$$\mathbf{K}_{ij}\frac{\partial \mathbf{U}}{\partial x_j} = \mathbf{E}_i. \tag{8}$$

Appropriate sets of boundary and initial conditions are assumed to accompany Eq. (6).

### 2.2 Incompressible flows

The Navier–Stokes equations of incompressible flows can be written on $\Omega$ and $\forall t \in (0, T)$ as

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} - \mathbf{f}\right) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} \;= 0, \tag{9}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} \;= 0, \tag{10}$$

where $\rho$ is assumed to be constant, and

$$\boldsymbol{\sigma}(p, \mathbf{u}) = -p\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}). \tag{11}$$

Here $\mathbf{I}$ is the identity tensor. An appropriate set of boundary conditions are assumed to accompany Eq. (9), and a divergence-free velocity field $\mathbf{u}_0(\mathbf{x})$ is specified as the initial condition.

## 3 STABILIZED FORMULATIONS

### 3.1 SUPG stabilization for compressible flows

Given Eq. (6), we form some suitably-defined finite-dimensional trial solution and test function spaces $\mathcal{S}^h_{\mathbf{U}}$ and $\mathcal{V}^h_{\mathbf{U}}$. The SUPG formulation of Eq. (6) can then be written as follows: find $\mathbf{U}^h \in \mathcal{S}^h_{\mathbf{U}}$ such that $\forall \mathbf{W}^h \in \mathcal{V}^h_{\mathbf{U}}$:

$$\begin{aligned}
& \int_\Omega \mathbf{W}^h \cdot \left(\frac{\partial \mathbf{U}^h}{\partial t} + \mathbf{A}^h_i \frac{\partial \mathbf{U}^h}{\partial x_i}\right) d\Omega + \int_\Omega \left(\frac{\partial \mathbf{W}^h}{\partial x_i}\right) \cdot \left(\mathbf{K}^h_{ij}\frac{\partial \mathbf{U}^h}{\partial x_j}\right) d\Omega \\
- & \int_{\Gamma_{\mathrm{H}}} \mathbf{W}^h \cdot \mathbf{H}^h d\Gamma - \int_\Omega \mathbf{W}^h \cdot \mathbf{R}^h d\Omega \\
+ & \sum_{e=1}^{n_{el}} \int_{\Omega^e} \boldsymbol{\tau}_{\mathrm{SUPG}} \left(\frac{\partial \mathbf{W}^h}{\partial x_k}\right) \cdot \mathbf{A}^h_k \left[\frac{\partial \mathbf{U}^h}{\partial t} + \mathbf{A}^h_i \frac{\partial \mathbf{U}^h}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\mathbf{K}^h_{ij}\frac{\partial \mathbf{U}^h}{\partial x_j}\right) - \mathbf{R}^h\right] d\Omega \\
+ & \sum_{e=1}^{n_{el}} \int_{\Omega^e} \nu_{\mathrm{SHOC}} \left(\frac{\partial \mathbf{W}^h}{\partial x_i}\right) \cdot \left(\frac{\partial \mathbf{U}^h}{\partial x_i}\right) d\Omega = 0 \; .
\end{aligned} \tag{12}$$

Here $\mathbf{H}^h$ represents the natural boundary conditions associated with Eq. (6), and $\Gamma_{\mathrm{H}}$ is the part of the boundary where such boundary conditions are specified. The SUPG stabilization and shock capturing parameters are denoted by $\boldsymbol{\tau}_{\mathrm{SUPG}}$ and $\nu_{\mathrm{SHOC}}$. They were discussed in Section 1 and will further be discussed in Section 6.

### 3.2 SUPG/PSPG stabilization for incompressible flows

Given Eqs. (9)–(10), we form some suitably-defined finite-dimensional trial solution and test function spaces for velocity and pressure: $\mathcal{S}_{\mathbf{u}}^h$, $\mathcal{V}_{\mathbf{u}}^h$, $\mathcal{S}_p^h$, and $\mathcal{V}_p^h = \mathcal{S}_p^h$. The stabilized finite element formulation of Eqs. (9)–(10) can then be written as follows: find $\mathbf{u}^h \in \mathcal{S}_{\mathbf{u}}^h$ and $p^h \in \mathcal{S}_p^h$ such that $\forall \mathbf{w}^h \in \mathcal{V}_{\mathbf{u}}^h$ and $\forall q^h \in \mathcal{V}_p^h$:

$$
\begin{aligned}
& \int_\Omega \mathbf{w}^h \cdot \rho \left( \frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h - \mathbf{f}^h \right) d\Omega + \int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}^h) : \boldsymbol{\sigma}(p^h, \mathbf{u}^h) d\Omega - \int_{\Gamma_\mathrm{h}} \mathbf{w}^h \cdot \mathbf{h}^h d\Gamma \\
+ & \int_\Omega q^h \boldsymbol{\nabla} \cdot \mathbf{u}^h d\Omega + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \frac{1}{\rho} \left[ \tau_{\mathrm{SUPG}} \rho \mathbf{u}^h \cdot \nabla \mathbf{w}^h + \tau_{\mathrm{PSPG}} \nabla q^h \right] \cdot \left[ \text{Ł}(p^h, \mathbf{u}^h) - \rho \mathbf{f}^h \right] d\Omega \\
+ & \sum_{e=1}^{n_{el}} \int_{\Omega^e} \nu_{\mathrm{LSIC}} \boldsymbol{\nabla} \cdot \mathbf{w}^h \rho \boldsymbol{\nabla} \cdot \mathbf{u}^h d\Omega \;=\; 0, \qquad (13)
\end{aligned}
$$

where

$$
\text{Ł}(q^h, \mathbf{w}^h) = \rho \left( \frac{\partial \mathbf{w}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{w}^h \right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(q^h, \mathbf{w}^h). \qquad (14)
$$

Here $\mathbf{h}^h$ represents the natural boundary conditions associated with Eq. (9), and $\Gamma_\mathrm{h}$ is the part of the boundary where such boundary conditions are specified. The SUPG, PSPG and LSIC (least-squares on incompressibility constraint) stabilization parameters are denoted by $\tau_{\mathrm{SUPG}}$, $\tau_{\mathrm{PSPG}}$ and $\nu_{\mathrm{LSIC}}$, respectively. They were discussed in Section 1 and will further be discussed in Section 4.

## 4 CALCULATION OF THE UGN/RGN-BASED STABILIZATION PARAMETERS

Various ways of calculating the stabilization parameters for incompressible flows were covered earlier in detail in [23, 13, 14, 15, 16, 17, 24, 18, 19]. In this section we focus on the versions of the stabilization parameters ($\tau$s) denoted by the subscript UGN, namely the UGN/RGN-based stabilization parameters. For this purpose, we first define the unit vectors $\mathbf{s}$ and $\mathbf{r}$:

$$
\mathbf{s} \;=\; \frac{\mathbf{u}^h}{\|\mathbf{u}^h\|}\,, \qquad (15)
$$

$$
\mathbf{r} \;=\; \frac{\boldsymbol{\nabla}\|\mathbf{u}^h\|}{\|\,\boldsymbol{\nabla}\|\mathbf{u}^h\|\,\|}\,. \qquad (16)
$$

The components of $(\tau_{\mathrm{SUPG}})_{\mathrm{UGN}}$ corresponding to the advection-, transient- and diffusion-dominated limits were defined in [15, 16, 17, 24, 18, 19] as follows:

$$\tau_{\text{SUGN1}} = \left( \sum_{a=1}^{n_{en}} |\mathbf{u}^h \cdot \boldsymbol{\nabla} N_a| \right)^{-1}, \tag{17}$$

$$\tau_{\text{SUGN2}} = \frac{\Delta t}{2}, \tag{18}$$

$$\tau_{\text{SUGN3}} = \frac{h_{\text{RGN}}^2}{4\nu}, \tag{19}$$

where $n_{en}$ is the number of element nodes and $N_a$ is the interpolation function associated with node $a$, and the "element length" $h_{\text{RGN}}$ is defined as

$$h_{\text{RGN}} = 2 \left( \sum_{a=1}^{n_{en}} |\mathbf{r} \cdot \boldsymbol{\nabla} N_a| \right)^{-1}. \tag{20}$$

Based on Eq. (17), the "element length" $h_{\text{UGN}}$ is defined as

$$h_{\text{UGN}} = 2\|\mathbf{u}^h\| \; \tau_{\text{SUGN1}} . \tag{21}$$

Although writing a direct expression for $\tau_{\text{SUGN1}}$ as given by Eq. (17) was pointed out in [15, 16, 17, 24, 18, 19], the element length definition one obtains by combining Eq. (17) and Eq. (21) was first introduced (as a direct expression for $h_{\text{UGN}}$) in [11]. The expression for $h_{\text{RGN}}$ as given by Eq. (20) was first introduced in [13, 14, 15]. It was noted in [15, 16, 17, 24, 18, 19] that $h_{\text{UGN}}$ and $h_{\text{RGN}}$ can be viewed as the local length scales corresponding to the advection- and diffusion-dominated limits, respectively.

We now define $(\tau_{\text{SUPG}})_{\text{UGN}}$, $(\tau_{\text{PSPG}})_{\text{UGN}}$, and $(\nu_{\text{LSIC}})_{\text{UGN}}$ as follows:

$$(\tau_{\text{SUPG}})_{\text{UGN}} = \left( \frac{1}{\tau_{\text{SUGN1}}^r} + \frac{1}{\tau_{\text{SUGN2}}^r} + \frac{1}{\tau_{\text{SUGN3}}^r} \right)^{-\frac{1}{r}}, \tag{22}$$

$$(\tau_{\text{PSPG}})_{\text{UGN}} = (\tau_{\text{SUPG}})_{\text{UGN}}, \tag{23}$$

$$(\nu_{\text{LSIC}})_{\text{UGN}} = (\tau_{\text{SUPG}})_{\text{UGN}} \; \|\mathbf{u}^h\|^2 . \tag{24}$$

Eq. (22) is based on the inverse of $(\tau_{\text{SUPG}})_{\text{UGN}}$ being defined as the $r$-norm of the vector with components $\frac{1}{\tau_{\text{SUGN1}}}$, $\frac{1}{\tau_{\text{SUGN2}}}$ and $\frac{1}{\tau_{\text{SUGN3}}}$. We note that the higher the integer $r$ is, the sharper the switching between $\tau_{\text{SUGN1}}$, $\tau_{\text{SUGN2}}$ and $\tau_{\text{SUGN3}}$ becomes. This "*r-switch*" was introduced in [23]. Typically, $r = 2$. The expressions for $\tau_{\text{SUGN3}}$ and $(\nu_{\text{LSIC}})_{\text{UGN}}$, given respectively by Eqs. (19) and (24), were proposed in [15, 16, 17, 24, 18, 19]. The "SUPG viscosity" $\nu_{\text{SUPG}}$ is defined as

$$\nu_{\text{SUPG}} = \tau_{\text{SUPG}} \|\mathbf{u}^h\|^2. \tag{25}$$

The space–time versions of $\tau_{\text{SUGN1}}$, $\tau_{\text{SUGN2}}$, $\tau_{\text{SUGN3}}$, $(\tau_{\text{SUPG}})_{\text{UGN}}$, $(\tau_{\text{PSPG}})_{\text{UGN}}$, and $(\nu_{\text{LSIC}})_{\text{UGN}}$, given respectively by Eqs. (17), (18), (19), (22), (23), and (24), were defined in [15, 16, 17, 24, 18, 19].

## 5 DISCONTINUITY-CAPTURING DIRECTIONAL DISSIPATION (DCDD)

As an alternative to the LSIC stabilization, the Discontinuity-Capturing Directional Dissipation (DCDD) stabilization was proposed in [13, 14, 15]. In describing the DCDD stabilization, we first define the "DCDD viscosity" $\nu_{\mathrm{DCDD}}$ and the DCDD stabilization parameter $\tau_{\mathrm{DCDD}}$:

$$\nu_{\mathrm{DCDD}} = \tau_{\mathrm{DCDD}} \|\mathbf{u}^h\|^2, \tag{26}$$

$$\tau_{\mathrm{DCDD}} = \frac{h_{\mathrm{DCDD}}}{2u_{\mathrm{ref}}} \frac{\| \boldsymbol{\nabla} \|\mathbf{u}^h\| \| \, h_{\mathrm{DCDD}}}{u_{\mathrm{ref}}} , \tag{27}$$

where

$$h_{\mathrm{DCDD}} = h_{\mathrm{RGN}} \ . \tag{28}$$

Here $u_{\mathrm{ref}}$ is a reference velocity (such as $\|\mathbf{u}^h\|$ at the inflow, or the difference between the estimated maximum and minimum values of $\|\mathbf{u}^h\|$). Combining Eqs. (26) and (27), we obtain

$$\nu_{\mathrm{DCDD}} = \frac{1}{2} \left( \frac{\|\mathbf{u}^h\|}{u_{\mathrm{ref}}} \right)^2 (h_{\mathrm{DCDD}})^2 \ \| \boldsymbol{\nabla} \|\mathbf{u}^h\| \ \| \ . \tag{29}$$

Then the DCDD stabilization is defined as

$$S_{\mathrm{DCDD}} = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \rho \boldsymbol{\nabla} \mathbf{w}^h : \left( [\nu_{\mathrm{DCDD}} \mathbf{r}\mathbf{r} - \boldsymbol{\kappa}_{\mathrm{CORR}}] \cdot \boldsymbol{\nabla} \mathbf{u}^h \right) d\Omega \ , \tag{30}$$

where $\boldsymbol{\kappa}_{\mathrm{CORR}}$ is defined as

$$\boldsymbol{\kappa}_{\mathrm{CORR}} = \nu_{\mathrm{DCDD}} (\mathbf{r} \cdot \mathbf{s})^2 \mathbf{s}\mathbf{s} \ . \tag{31}$$

## 6 CALCULATION OF THE STABILIZATION PARAMETERS FOR COMPRESSIBLE FLOWS AND SHOCK-CAPTURING

The SUPG formulation for compressible flows was first introduced, in the context of conservation variables, in [3, 4, 5]. Here we will call that formulation "$(SUPG)_{82}$". In this section, in the context of the $(SUPG)_{82}$ formulation and based on the ideas we discussed in Sections 4 and 5, we propose alternative ways of calculating the stabilization parameters and defining the shock-capturing terms. For this purpose, we first define the acoustic speed as $c$, and define the unit vector $\mathbf{j}$ as

$$\mathbf{j} = \frac{\boldsymbol{\nabla} \rho^h}{\| \boldsymbol{\nabla} \rho^h \|} \ . \tag{32}$$

As the first alternative in computing $\tau_{\mathrm{SUGN1}}$ for each component of the test vector-function $\mathbf{W}$, we propose to define ${\tau^{\rho}}_{\mathrm{SUGN1}}$, ${\tau^{u}}_{\mathrm{SUGN1}}$ and ${\tau^{e}}_{\mathrm{SUGN1}}$ (associated with $\rho$, $\rho\mathbf{u}$ and $\rho e$, respectively) by using the expression given by Eq. (17):

$$ {\tau^{\rho}}_{\mathrm{SUGN1}} = {\tau^{u}}_{\mathrm{SUGN1}} = {\tau^{e}}_{\mathrm{SUGN1}} = \left( \sum_{a=1}^{n_{en}} |\mathbf{u}^h \cdot \boldsymbol{\nabla} N_a| \right)^{-1} . \tag{33} $$

As the second alternative, we propose to use the following definition:

$$ {\tau^{\rho}}_{\mathrm{SUGN1}} = {\tau^{u}}_{\mathrm{SUGN1}} = {\tau^{e}}_{\mathrm{SUGN1}} = \left( \sum_{a=1}^{n_{en}} \left( c \, |\mathbf{j} \cdot \boldsymbol{\nabla} N_a| + |\mathbf{u}^h \cdot \boldsymbol{\nabla} N_a| \right) \right)^{-1} . \tag{34} $$

In computing $\tau_{\mathrm{SUGN2}}$, we propose to use the expression given by Eq. (18):

$$ {\tau^{\rho}}_{\mathrm{SUGN2}} = {\tau^{u}}_{\mathrm{SUGN2}} = {\tau^{e}}_{\mathrm{SUGN2}} = \frac{\Delta t}{2} . \tag{35} $$

In computing $\tau_{\mathrm{SUGN3}}$, we propose to define ${\tau^{u}}_{\mathrm{SUGN3}}$ by using the expression given by Eq. (19):

$$ {\tau^{u}}_{\mathrm{SUGN3}} = \frac{h_{\mathrm{RGN}}^2}{4\nu} . \tag{36} $$

We propose to define ${\tau^{e}}_{\mathrm{SUGN3}}$ as

$$ {\tau^{e}}_{\mathrm{SUGN3}} = \frac{({h^{e}}_{\mathrm{RGN}})^2}{4\nu^e} , \tag{37} $$

where $\nu^e$ is the "kinematic viscosity" for the energy equation,

$$ {h^{e}}_{\mathrm{RGN}} = 2 \left( \sum_{a=1}^{n_{en}} |\mathbf{r}^e \cdot \boldsymbol{\nabla} N_a| \right)^{-1} , \tag{38} $$

$$ \mathbf{r}^e = \frac{\boldsymbol{\nabla}\theta^h}{\| \boldsymbol{\nabla}\theta^h \|} , \tag{39} $$

and $\theta$ is the temperature. We define $(\tau^{\rho}_{\mathrm{SUPG}})_{\mathrm{UGN}}$, $(\tau^{u}_{\mathrm{SUPG}})_{\mathrm{UGN}}$ and $(\tau^{e}_{\mathrm{SUPG}})_{\mathrm{UGN}}$ by using the "*r-switch*" given in Section 4 :

$$ (\tau^{\rho}_{\mathrm{SUPG}})_{\mathrm{UGN}} = \left( \frac{1}{({\tau^{\rho}}_{\mathrm{SUGN1}})^r} + \frac{1}{({\tau^{\rho}}_{\mathrm{SUGN2}})^r} \right)^{-\frac{1}{r}} , \tag{40} $$

$$ (\tau^{u}_{\mathrm{SUPG}})_{\mathrm{UGN}} = \left( \frac{1}{({\tau^{u}}_{\mathrm{SUGN1}})^r} + \frac{1}{({\tau^{u}}_{\mathrm{SUGN2}})^r} + \frac{1}{({\tau^{u}}_{\mathrm{SUGN3}})^r} \right)^{-\frac{1}{r}} , \tag{41} $$

$$ (\tau^{e}_{\mathrm{SUPG}})_{\mathrm{UGN}} = \left( \frac{1}{({\tau^{e}}_{\mathrm{SUGN1}})^r} + \frac{1}{({\tau^{e}}_{\mathrm{SUGN2}})^r} + \frac{1}{({\tau^{e}}_{\mathrm{SUGN3}})^r} \right)^{-\frac{1}{r}} . \tag{42} $$

In defining the shock-capturing term, we first define the "shock-capturing viscosity" $\nu_{\text{SHOC}}$:

$$\nu_{\text{SHOC}} = \tau_{\text{SHOC}} (u_{\text{int}})^2, \tag{43}$$

where

$$\tau_{\text{SHOC}} = \frac{h_{\text{SHOC}}}{2u_{\text{cha}}} \left( \frac{\| \nabla \rho^h \| \, h_{\text{SHOC}}}{\rho_{\text{ref}}} \right)^{\beta}, \tag{44}$$

$$h_{\text{SHOC}} = h_{\text{JGN}} \,, \tag{45}$$

$$h_{\text{JGN}} = 2 \left( \sum_{a=1}^{n_{en}} |\mathbf{j} \cdot \nabla N_a| \right)^{-1}. \tag{46}$$

Here $\rho_{\text{ref}}$ is a reference density (such as $\rho^h$ at the inflow, or the difference between the estimated maximum and minimum values of $\rho^h$), $u_{\text{cha}}$ is a characteristic velocity (such as $u_{\text{ref}}$ or $\|\mathbf{u}^h\|$ or acoustic speed $c$), and $u_{\text{int}}$ is an intrinsic velocity (such as $u_{\text{cha}}$ or $\|\mathbf{u}^h\|$ or acoustic speed $c$). We propose to set $u_{\text{int}} = u_{\text{cha}} = u_{\text{ref}}$. The parameter $\beta$ influences the smoothness of the shock-front. We set $\beta = 1$ for smoother shocks and $\beta = 2$ for sharper shocks (in return for tolerating possible overshoots and undershoots). As a compromise between the $\beta = 1$ and $\beta = 2$ selections, we propose the following averaged expression for $\tau_{\text{SHOC}}$ :

$$\tau_{\text{SHOC}} = \frac{1}{2}\Big( (\tau_{\text{SHOC}})_{\beta=1} + (\tau_{\text{SHOC}})_{\beta=2} \Big) \,. \tag{47}$$

As an alternate way, we also propose to calculate $\nu_{\text{SHOC}}$ by using the following expression:

$$\nu_{\text{SHOC}} = \left\| \mathbf{Y}^{-1}\mathbf{Z} \right\| \left( \sum_{i=1}^{n_{sd}} \left\| \mathbf{Y}^{-1} \frac{\partial \mathbf{U}^h}{\partial x_i} \right\|^2 \right)^{\beta/2-1} \left( \frac{h_{\text{SHOC}}}{2} \right)^{\beta}, \tag{48}$$

where $\mathbf{Y}$ is a diagonal scaling matrix constructed from the reference values of the components of $\mathbf{U}$ :

$$\mathbf{Y} = \begin{bmatrix} (U_1)_{\text{ref}} & 0 & 0 & 0 & 0 \\ 0 & (U_2)_{\text{ref}} & 0 & 0 & 0 \\ 0 & 0 & (U_3)_{\text{ref}} & 0 & 0 \\ 0 & 0 & 0 & (U_4)_{\text{ref}} & 0 \\ 0 & 0 & 0 & 0 & (U_5)_{\text{ref}} \end{bmatrix}, \tag{49}$$

$$\mathbf{Z} \;=\; \frac{\partial \mathbf{U}^h}{\partial t} + \mathbf{A}_i^h \frac{\partial \mathbf{U}^h}{\partial x_i} \tag{50}$$

$$OR$$

$$\mathbf{Z} \;=\; \mathbf{A}_i^h \frac{\partial \mathbf{U}^h}{\partial x_i}\ , \tag{51}$$

and we set $\beta = 1$ or $\beta = 2$. As a variation of the expression given by Eq. 48, we propose for $\nu_{\rm SHOC}$ the following expression:

$$\nu_{\rm SHOC} = \left\| \mathbf{Y}^{-1}\mathbf{Z} \right\| \left( \sum_{i=1}^{n_{sd}} \left\| \mathbf{Y}^{-1} \frac{\partial \mathbf{U}^h}{\partial x_i} \right\|^2 \right)^{\beta/2-1} \left\| \mathbf{Y}^{-1}\mathbf{U}^h \right\|^{1-\beta} \left( \frac{h_{\rm SHOC}}{2} \right)^{\beta} . \tag{52}$$

As a compromise between the $\beta = 1$ and $\beta = 2$ selections, we propose the following averaged expression for $\nu_{\rm SHOC}$ :

$$\nu_{\rm SHOC} = \frac{1}{2}\Big( (\nu_{\rm SHOC})_{\beta=1} + (\nu_{\rm SHOC})_{\beta=2} \Big) \ . \tag{53}$$

We can also calculate, based on Eq. 48, a separate $\nu_{\rm SHOC}$ for each component of the test vector-function $\mathbf{W}$ :

$$(\nu_{\rm SHOC})_{\rm I} = \left| \left( \mathbf{Y}^{-1}\mathbf{Z} \right)_{\rm I} \right| \left( \sum_{i=1}^{n_{sd}} \left| \left( \mathbf{Y}^{-1} \frac{\partial \mathbf{U}^h}{\partial x_i} \right)_{\rm I} \right|^2 \right)^{\beta/2-1} \left( \frac{h_{\rm SHOC}}{2} \right)^{\beta}, \quad I = 1, 2, \ldots n_{sd} + 2 \ . \tag{54}$$

Similarly, a separate $\nu_{\rm SHOC}$ for each component of $\mathbf{W}$ can be calculated based on Eq. 52 :

$$(\nu_{\rm SHOC})_{\rm I} = \left| \left( \mathbf{Y}^{-1}\mathbf{Z} \right)_{\rm I} \right| \left( \sum_{i=1}^{n_{sd}} \left| \left( \mathbf{Y}^{-1} \frac{\partial \mathbf{U}^h}{\partial x_i} \right)_{\rm I} \right|^2 \right)^{\beta/2-1} \left| \left( \mathbf{Y}^{-1}\mathbf{U}^h \right)_{\rm I} \right|^{1-\beta} \left( \frac{h_{\rm SHOC}}{2} \right)^{\beta}, \quad I = 1, 2, \ldots n_{sd} + 2 \ . \tag{55}$$

Given $\nu_{\rm SHOC}$, the shock-capturing term is defined as

$$S_{\rm SHOC} = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \boldsymbol{\nabla}\mathbf{W}^h : \left( \boldsymbol{\kappa}_{\rm SHOC} \cdot \boldsymbol{\nabla}\mathbf{U}^h \right) d\Omega \ , \tag{56}$$

where $\boldsymbol{\kappa}_{\rm SHOC}$ is defined as

$$\boldsymbol{\kappa}_{\rm SHOC} = \nu_{\rm SHOC}\ \mathbf{I} \ . \tag{57}$$

As a possible alternative, we propose

$$\boldsymbol{\kappa}_{\rm SHOC} = \nu_{\rm SHOC}\ \mathbf{jj} \ . \tag{58}$$

If the option given by Eq. 54 or Eq. 55 is exercised, then $\nu_{\mathrm{SHOC}}$ becomes an $(n_{sd}+2)\times(n_{sd}+2)$ diagonal matrix, and the matrix $\boldsymbol{\kappa}_{\mathrm{SHOC}}$ becomes augmented from an $n_{sd}\times n_{sd}$ matrix to an $(n_{sd}\times(n_{sd}+2))\times((n_{sd}+2)\times n_{sd})$ matrix.

In an attempt to preclude compounding, we propose to modify $\nu_{\mathrm{SHOC}}$ as follows:

$$\nu_{\mathrm{SHOC}} \;\leftarrow\; \nu_{\mathrm{SHOC}} - switch\left(\tau_{\mathrm{SUPG}}\left(\mathbf{j}\cdot\mathbf{u}\right)^2,\ \tau_{\mathrm{SUPG}}\left(|\mathbf{j}\cdot\mathbf{u}|-c\right)^2,\ \nu_{\mathrm{SHOC}}\right)\ , \tag{59}$$

where the "*switch*" function is defined as the "*min*" function or as the "*r-switch*" given in Section 4. For viscous flows, the above modification would be made separately with each of $\tau^{\rho}_{\mathrm{SUPG}}$, $\tau^{u}_{\mathrm{SUPG}}$ and $\tau^{e}_{\mathrm{SUPG}}$, and this would result in $\nu_{\mathrm{SHOC}}$ becoming a diagonal matrix even if the option given by Eq. 54 or Eq. 55 is not exercised.

## 7 CONCLUDING REMARKS

We described, for the Streamline-Upwind/Petrov-Galerkin (SUPG) formulation of compressible flows based on conservation variables, new ways for determining the stabilization and shock-capturing parameters. The stabilization parameter, which is typically known as "$\tau$", plays an important role in determining the accuracy of the solutions. The shock-capturing term provides additional stabilization near the shocks, and how the shock-capturing parameter it involves is defined influences the quality of the solution near the shocks. These new ways of calculating the $\tau$s and shock-capturing parameters are partly based on the ideas underlying the $\tau$s and and DCDD stabilization developed for incompressible flows. Compared to the earlier shock-capturing parameter that was derived based on the entropy variables, the new one is much simpler and computationally less costly.

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## REFERENCES

[1] T.J.R. Hughes and A.N. Brooks, “A multi-dimensional upwind scheme with no cross-wind diffusion”, in T.J.R. Hughes, editor, *Finite Element Methods for Convection Dominated Flows*, AMD-Vol.34, 19–35, ASME, New York, 1979.

[2] A.N. Brooks and T.J.R. Hughes, “Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations”, *Computer Methods in Applied Mechanics and Engineering*, **32** (1982) 199–259.

[3] T.E. Tezduyar and T.J.R. Hughes, “Development of time-accurate finite element techniques for first-order hyperbolic systems with particular emphasis on the compressible Euler equations”, NASA Technical Report NASA-CR-204772, NASA, 1982, http://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19970023187_1997034954.pdf.

[4] T.E. Tezduyar and T.J.R. Hughes, “Finite element formulations for convection dominated flows with particular emphasis on the compressible Euler equations”, in *Proceedings of AIAA 21st Aerospace Sciences Meeting*, AIAA Paper 83-0125, Reno, Nevada, (1983).

[5] T.J.R Hughes and T.E. Tezduyar, “Finite element methods for first-order hyperbolic systems with particular emphasis on the compressible Euler equations”, *Computer Methods in Applied Mechanics and Engineering*, **45** (1984) 217–284.

[6] T.E. Tezduyar, “Stabilized finite element formulations for incompressible flow computations”, *Advances in Applied Mechanics*, **28** (1992) 1–44.

[7] J. Donea, “A Taylor-Galerkin method for convective transport problems”, *International Journal for Numerical Methods in Engineering*, **20** (1984) 101–120.

[8] C. Johnson, U. Navert, and J. Pitkäranta, “Finite element methods for linear hyperbolic problems”, *Computer Methods in Applied Mechanics and Engineering*, **45** (1984) 285–312.

[9] T.J.R. Hughes, L.P. Franca, and M. Mallet, “A new finite element formulation for computational fluid dynamics: VI. Convergence analysis of the generalized SUPG formulation for linear time-dependent multi-dimensional advective-diffusive systems”, *Computer Methods in Applied Mechanics and Engineering*, **63** (1987) 97–112.

[10] G.J. Le Beau and T.E. Tezduyar, “Finite element computation of compressible flows with the SUPG formulation”, in *Advances in Finite Element Analysis in Fluid Dynamics*, FED-Vol.123, ASME, New York, (1991) 21–27.

[11] T.E. Tezduyar and Y.J. Park, "Discontinuity capturing finite element formulations for nonlinear convection-diffusion-reaction equations", *Computer Methods in Applied Mechanics and Engineering*, **59** (1986) 307–325.

[12] T.J.R. Hughes, L.P. Franca, and M. Balestra, "A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuška–Brezzi condition: A stable Petrov–Galerkin formulation of the Stokes problem accommodating equal-order interpolations", *Computer Methods in Applied Mechanics and Engineering*, **59** (1986) 85–99.

[13] T.E. Tezduyar, "Adaptive determination of the finite element stabilization parameters", in *Proceedings of the ECCOMAS Computational Fluid Dynamics Conference 2001 (CD-ROM)*, Swansea, Wales, United Kingdom, (2001).

[14] T.E. Tezduyar, "Stabilized finite element formulations and interface-tracking and interface-capturing techniques for incompressible flows", in M.M. Hafez, editor, *Numerical Simulations of Incompressible Flows*, World Scientific, New Jersey, (2003) 221–239.

[15] T.E. Tezduyar, "Computation of moving boundaries and interfaces and stabilization parameters", *International Journal for Numerical Methods in Fluids*, **43** (2003) 555–575.

[16] T. Tezduyar, "Stabilization parameters and local length scales in SUPG and PSPG formulations", in *Proceedings of the Fifth World Congress on Computational Mechanics*, On-line publication: http://wccm.tuwien.ac.at/, Paper-ID: 81508, Vienna, Austria, (2002).

[17] T. Tezduyar and S. Sathe, "Stabilization parameters in SUPG and PSPG formulations", *Journal of Computational and Applied Mechanics*, **4** (2003) 71–88.

[18] T.E. Tezduyar, "Finite element methods for fluid dynamics with moving boundaries and interfaces", in E. Stein, R. De Borst, and T.J.R. Hughes, editors, *Encyclopedia of Computational Mechanics*, Volume 3: Fluids, Chapter 17, John Wiley & Sons, 2004.

[19] T.E. Tezduyar, "Calculation of the stabilization parameters in finite element formulations of flow problems", in S.R. Idelsohn and V. Sonzogni, editors, *Applications of Computational Mechanics in Structures and Fluids*, 1–19, CIMNE, Barcelona, Spain, 2005.

[20] T.E. Tezduyar, "Finite element computational methods for moving boundaries and interfaces", in *Proceedings of the Sixth Hellenic-European Conference on Computer Mathematics and its Applications*, Athens, Greece, (2003).

[21] T.E. Tezduyar, "Stabilized finite element methods for computation of flows with moving boundaries and interfaces", in *Lecture Notes on Finite Element Simulation of Flow Problems (Basic - Advanced Course)*, Japan Society of Computational Engineering and Sciences, Tokyo, Japan, (2003).

[22] T.E. Tezduyar, "Stabilized finite element methods for flows with moving boundaries and interfaces", *HERMIS: The International Journal of Computer Mathematics and its Applications*, **4** (2003) 63–88.

[23] T.E. Tezduyar and Y. Osawa, "Finite element stabilization parameters computed from element matrices and vectors", *Computer Methods in Applied Mechanics and Engineering*, **190** (2000) 411–430.

[24] T.E. Tezduyar, "Calculation of the stabilization parameters in SUPG and PSPG formulations", in *Proceedings of the First South-American Congress on Computational Mechanics (CD-ROM)*, Santa Fe–Parana, Argentina, (2002).