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Stabilization and shock-capturing parameters in SUPG formulation of compressible flows

Tayfun E. Tezduyar *, Masayoshi Senga

Mechanical Engineering, Rice University—MS 321, 6100 Main Street, Houston, TX 77005-1892, USA

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Abstract

The streamline-upwind/Petrov–Galerkin (SUPG) formulation is one of the most widely used stabilized methods in finite element computation of compressible flows. It includes a stabilization parameter that is known as "τ". Typically the SUPG formulation is used in combination with a shock-capturing term that provides additional stability near the shock fronts. The definition of the shock-capturing term includes a shock-capturing parameter. In this paper, we describe, for the finite element formulation of compressible flows based on conservation variables, new ways for determining the τ and the shock-capturing parameter. The new definitions for the shock-capturing parameter are much simpler than the one based on the entropy variables, involve less operations in calculating the shock-capturing term, and yield better shock quality in the test computations.

Keywords: Compressible flows; Finite element formulation; SUPG stabilization, Stabilization parameters; Shock-capturing parameter

1. Introduction

In finite element computation of flow problems, the streamline-upwind/Petrov–Galerkin (SUPG) formulation for incompressible flows [1,2], the SUPG formulation for compressible flows [3–5], and the pressure-stabilizing/Petrov–Galerkin (PSPG) formulation for incompressible flows [6] are some of the most prevalent stabilized methods. Stabilized formulations such as the SUPG and PSPG formulations prevent numerical instabilities in solving problems with high Reynolds or Mach numbers and shocks or thin boundary layers, as well as when using equal-order interpolation functions for velocity and pressure.

* Corresponding author. Tel.: +1 713 348 6051; fax: +1 713 348 5423.
E-mail addresses: tezduyar@rice.edu (T.E. Tezduyar), msenga@rice.edu (M. Senga).

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The SUPG formulation for incompressible flows was first introduced in [1], with further studies in [2]. The SUPG formulation for compressible flows was first introduced, in the context of conservation variables in [3]. A concise version of that was published as an AIAA paper [4], and a more thorough version with additional examples as a journal paper [5]. After that, several SUPG-like methods for compressible flows were developed. Taylor–Galerkin method [7], for example, is very similar, and under certain conditions is identical, to one of the SUPG methods introduced in [3–5]. Later, following [3–5], the SUPG formulation for compressible flows was recast in entropy variables and supplemented with a shock-capturing term [8]. It was shown in [9] that the SUPG formulation introduced in [3–5], when supplemented with a similar shock-capturing term, is very comparable in accuracy to the one that was recast in entropy variables. The stabilized formulation introduced in [10] for advection–diffusion–reaction equations also included a shock-capturing (discontinuity-capturing) term, and precluded augmentation of the SUPG effect by the discontinuity-capturing effect when the advection and discontinuity directions coincide.

A stabilization parameter, known as "τ", is embedded in the SUPG and PSPG formulations. It involves a measure of the local length scale (also known as "element length") and other parameters such as the element Reynolds and Courant numbers. Various element lengths and τs were proposed starting with those in [1–5], followed by the one introduced in [10], and those proposed in the subsequently reported SUPG-based methods. Here we will call the SUPG formulation introduced in [3–5] for compressible flows "$(\mathrm{SUPG})_{82}$", and the set of τs introduced in conjunction with that "τ_{82}". The τ used in [9] with $(\mathrm{SUPG})_{82}$ is a slightly modified version of τ_{82}. A shock-capturing parameter, which we will call here "δ_{91}", was embedded in the shock-capturing term used in [9]. Subsequent minor modifications of τ_{82} took into account the interaction between the shock-capturing and the $(\mathrm{SUPG})_{82}$ terms in a fashion similar to how it was done in [10] for advection–diffusion–reaction equations. All these slightly modified versions of τ_{82} have always been used with the same δ_{91}, and we will categorize them here all under the label "$\tau_{82\text{-MOD}}$".

To be used in conjunction with the SUPG/PSPG formulation of incompressible flows, the discontinuity-capturing directional dissipation (DCDD) stabilization was introduced in [11,12] for flow fields with sharp gradients. This involved a second element length scale, which was also introduced in [11,12] and is based on the solution gradient. This new element length scale is used together with the element length scales already defined in [10]. Recognizing this second element length as a diffusion length scale, new stabilization parameters for the diffusive limit were introduced in [12–14]. The DCDD stabilization was originally conceived in [11,12] as an alternative to the LSIC (least-squares on incompressibility constraint) stabilization. The DCDD takes effect where there is a sharp gradient in the velocity field and introduces dissipation in the direction of that gradient. The way the DCDD is added to the formulation precludes augmentation of the SUPG effect by the DCDD effect when the advection and discontinuity directions coincide.

Partly based on the ideas underlying the new τs for incompressible flows and the DCDD, new ways of calculating the τs and shock-capturing parameters for compressible flows were introduced in [14–17]. Like the parameters developed earlier, these new parameters are intended for use with the SUPG formulation of compressible flows based on conservation variables. In this paper, we describe how the new parameters are defined.

2. Navier–Stokes equations of compressible flows

Let $\Omega \subset \mathbb{R}^{n_{\mathrm{sd}}}$ be the spatial domain with boundary Γ, and $(0, T)$ be the time domain. The symbols ρ, $\mathbf{u}$, p and e will represent the density, velocity, pressure and the total energy, respectively.

The Navier–Stokes equations of compressible flows can be written on Ω and $\forall t \in (0, T)$ as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} - \frac{\partial \mathbf{E}_i}{\partial x_i} - \mathbf{R} = \mathbf{0}, \tag{1}$$

where $\mathbf{U} = (\rho, \rho u_1, \rho u_2, \rho u_3, \rho e)$ is the vector of conservation variables, and $\mathbf{F}_i$ and $\mathbf{E}_i$ are, respectively, the Euler and viscous flux vectors:

$$\mathbf{F}_i = \begin{pmatrix} u_i\rho \\ u_i\rho u_1 + \delta_{i1}p \\ u_i\rho u_2 + \delta_{i2}p \\ u_i\rho u_3 + \delta_{i3}p \\ u_i(\rho e + p) \end{pmatrix}, \quad \mathbf{E}_i = \begin{pmatrix} 0 \\ T_{i1} \\ T_{i2} \\ T_{i3} \\ -q_i + T_{ik}u_k \end{pmatrix}. \tag{2}$$

Here δ_{ij} are the components of the identity tensor $\mathbf{I}$, q_i are the components of the heat flux vector, and T_{ij} are the components of the Newtonian viscous stress tensor:

$$\mathbf{T} = \lambda(\nabla \cdot \mathbf{u})\mathbf{I} + 2\mu\varepsilon(\mathbf{u}), \tag{3}$$

where λ and μ $(=\rho\nu)$ are the viscosity coefficients, ν is the kinematic viscosity, and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the strain-rate tensor:

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}((\nabla\mathbf{u}) + (\nabla\mathbf{u})^{\mathrm{T}}). \tag{4}$$

It is assumed that $\lambda = -2\mu/3$. The equation of state used here corresponds to the ideal gas assumption. The term $\mathbf{R}$ represents all other components that might enter the equations, including the external forces.

Eq. (1) can further be written in the following form:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}_i \frac{\partial \mathbf{U}}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\mathbf{K}_{ij}\frac{\partial \mathbf{U}}{\partial x_j}\right) - \mathbf{R} = \mathbf{0}, \tag{5}$$

where

$$\mathbf{A}_i = \frac{\partial \mathbf{F}_i}{\partial \mathbf{U}}, \quad \mathbf{K}_{ij}\frac{\partial \mathbf{U}}{\partial x_j} = \mathbf{E}_i. \tag{6}$$

Appropriate sets of boundary and initial conditions are assumed to accompany Eq. (5).

3. SUPG formulations

3.1. Semi-discrete

Given Eq. (5), we form some suitably-defined finite-dimensional trial solution and test function spaces $\mathcal{S}^h_{\mathbf{U}}$ and $\mathcal{V}^h_{\mathbf{U}}$. The SUPG formulation of Eq. (5) can then be written as follows: find $\mathbf{U}^h \in \mathcal{S}^h_{\mathbf{U}}$ such that $\forall \mathbf{W}^h \in \mathcal{V}^h_{\mathbf{U}}$:

$$\begin{aligned} &\int_\Omega \mathbf{W}^h \cdot \left(\frac{\partial \mathbf{U}^h}{\partial t} + \mathbf{A}^h_i \frac{\partial \mathbf{U}^h}{\partial x_i}\right) \mathrm{d}\Omega + \int_\Omega \left(\frac{\partial \mathbf{W}^h}{\partial x_i}\right) \cdot \left(\mathbf{K}^h_{ij}\frac{\partial \mathbf{U}^h}{\partial x_j}\right) \mathrm{d}\Omega - \int_{\Gamma_{\mathrm{H}}} \mathbf{W}^h \cdot \mathbf{H}^h \,\mathrm{d}\Gamma \\ &\quad - \int_\Omega \mathbf{W}^h \cdot \mathbf{R}^h \,\mathrm{d}\Omega + \sum_{e=1}^{n_{\mathrm{el}}} \int_{\Omega^e} \boldsymbol{\tau}_{\mathrm{SUPG}} \left(\frac{\partial \mathbf{W}^h}{\partial x_k}\right) \cdot \mathbf{A}^h_k \left[\frac{\partial \mathbf{U}^h}{\partial t} + \mathbf{A}^h_i \frac{\partial \mathbf{U}^h}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\mathbf{K}^h_{ij}\frac{\partial \mathbf{U}^h}{\partial x_j}\right) - \mathbf{R}^h\right] \mathrm{d}\Omega \\ &\quad + \sum_{e=1}^{n_{\mathrm{el}}} \int_{\Omega^e} \nu_{\mathrm{SHOC}} \left(\frac{\partial \mathbf{W}^h}{\partial x_i}\right) \cdot \left(\frac{\partial \mathbf{U}^h}{\partial x_i}\right) \mathrm{d}\Omega = 0. \end{aligned} \tag{7}$$

Here $\mathbf{H}^h$ represents the natural boundary conditions associated with Eq. (5), and Γ_{H} is the part of the boundary where such boundary conditions are specified. The SUPG stabilization and shock-capturing parameters are denoted by $\boldsymbol{\tau}_{\mathrm{SUPG}}$ and ν_{SHOC}. They were discussed in Section 1 and will further be discussed in Section 4.

3.2. Space–time

The space–time version of Eq. (7) can be written based on the deforming-spatial-domain/stabilized space–time (DSD/SST) formulation introduced in [6,18,19]. The finite element formulation of the governing equations is written over a sequence of N space–time slabs Q_n, where Q_n is the slice of the space–time domain between the time levels t_n and t_{n+1}. At each time step, the integrations involved in the finite element formulation are performed over Q_n. The finite element interpolation functions are discontinuous across the space–time slabs. We use the notation $(\cdot)_n^-$ and $(\cdot)_n^+$ to denote the values as t_n is approached from below and above respectively. Each Q_n is decomposed into space–time elements Q_n^e, where $e = 1, 2, \ldots, (n_{\mathrm{el}})_n$. The subscript n used with n_{el} is to account for the general case in which the number of space–time elements may change from one space–time slab to another. For each slab Q_n, we form some suitably-defined finite-dimensional trial solution and test function spaces $(\mathscr{S}_{\mathrm{U}}^h)_n$ and $(\mathscr{V}_{\mathrm{U}}^h)_n$. In the computations reported here, we use first-order polynomials as interpolation functions. The subscript n implies that corresponding to different space–time slabs we might have different discretizations. The DSD/SST formulation of Eq. (5) can then be written as follows: given $(\mathbf{U}^h)_n^-$, find $\mathbf{U}^h \in (\mathscr{S}_{\mathrm{U}}^h)_n$ such that $\forall \mathbf{W}^h \in (\mathscr{V}_{\mathrm{U}}^h)_n$:

$$
\begin{aligned}
&\int_{Q_n} \mathbf{W}^h \cdot \left(\frac{\partial \mathbf{U}^h}{\partial t} + \mathbf{A}_i^h \frac{\partial \mathbf{U}^h}{\partial x_i} \right) \mathrm{d}Q + \int_{Q_n} \left(\frac{\partial \mathbf{W}^h}{\partial x_i} \right) \cdot \left(\mathbf{K}_{ij}^h \frac{\partial \mathbf{U}^h}{\partial x_j} \right) \mathrm{d}Q - \int_{P_n} \mathbf{W}^h \cdot \mathbf{H}^h \, \mathrm{d}P \\
&\quad - \int_{Q_n} \mathbf{W}^h \cdot \mathbf{R}^h \, \mathrm{d}Q + \int_{\Omega} (\mathbf{W}^h)_n^+ \cdot \left((\mathbf{U}^h)_n^+ - (\mathbf{U}^h)_n^- \right) \mathrm{d}\Omega \\
&\quad + \sum_{e=1}^{(n_{\mathrm{el}})_n} \int_{Q_n^e} \boldsymbol{\tau}_{\mathrm{SUPG}} \left(\frac{\partial \mathbf{W}^h}{\partial x_k} \right) \cdot \mathbf{A}_k^h \left[\frac{\partial \mathbf{U}^h}{\partial t} + \mathbf{A}_i^h \frac{\partial \mathbf{U}^h}{\partial x_i} - \frac{\partial}{\partial x_i} \left(\mathbf{K}_{ij}^h \frac{\partial \mathbf{U}^h}{\partial x_j} \right) - \mathbf{R}^h \right] \mathrm{d}Q \\
&\quad + \sum_{e=1}^{(n_{\mathrm{el}})_n} \int_{Q_n^e} \nu_{\mathrm{SHOC}} \left(\frac{\partial \mathbf{W}^h}{\partial x_i} \right) \cdot \left(\frac{\partial \mathbf{U}^h}{\partial x_i} \right) \mathrm{d}Q = 0.
\end{aligned} \tag{8}
$$

Here P_n is the lateral boundary of the space–time slab. The solution to Eq. (8) is obtained sequentially for all space–time slabs $Q_0, Q_1, Q_2, \ldots, Q_{N-1}$, and the computations start with $(\mathbf{U}^h)_0^- = \mathbf{U}_0^h$, where $\mathbf{U}_0$ is the specified initial value of the vector $\mathbf{U}$.

4. Calculation of the stabilization parameters for compressible flows and shock-capturing

Various options for calculating the stabilization parameters and defining the shock-capturing terms in the context of the $(\mathrm{SUPG})_{82}$ formulation were introduced in [14–17]. In this section we describe those options. For this purpose, we first define the acoustic speed as c, and define the unit vector $\mathbf{j}$ as

$$
\mathbf{j} = \frac{\boldsymbol{\nabla}\rho^h}{\|\boldsymbol{\nabla}\rho^h\|}. \tag{9}
$$

As the first option in computing τ_{SUGN1} for each component of the test vector-function $\mathbf{W}$, the stabilization parameters $\tau_{\mathrm{SUGN1}}^{\rho}$, $\tau_{\mathrm{SUGN1}}^{u}$ and $\tau_{\mathrm{SUGN1}}^{e}$ (associated with ρ, $\rho\mathbf{u}$ and ρe, respectively) are defined by the following expression:

$$
\tau_{\mathrm{SUGN1}}^{\rho} = \tau_{\mathrm{SUGN1}}^{u} = \tau_{\mathrm{SUGN1}}^{e} = \left(\sum_{a=1}^{n_{en}} |\mathbf{u}^h \cdot \boldsymbol{\nabla} N_a| \right)^{-1}. \tag{10}
$$

As the second option, they are defined as

$$\tau^{\rho}_{\mathrm{SUGN1}} = \tau^{u}_{\mathrm{SUGN1}} = \tau^{e}_{\mathrm{SUGN1}} = \left(\sum_{a=1}^{n_{en}} (c|\mathbf{j} \cdot \nabla N_a| + |\mathbf{u}^h \cdot \nabla N_a|) \right)^{-1}. \tag{11}$$

In computing τ_{SUGN2}, the parameters $\tau^{\rho}_{\mathrm{SUGN2}}$, $\tau^{u}_{\mathrm{SUGN2}}$ and $\tau^{e}_{\mathrm{SUGN2}}$ are defined as follows:

$$\tau^{\rho}_{\mathrm{SUGN2}} = \tau^{u}_{\mathrm{SUGN2}} = \tau^{e}_{\mathrm{SUGN2}} = \frac{\Delta t}{2}, \tag{12}$$

where Δt is the time step. In computing τ_{SUGN3}, the parameter $\tau^{u}_{\mathrm{SUGN3}}$ is defined by using the expression

$$\tau^{u}_{\mathrm{SUGN3}} = \frac{h^2_{\mathrm{RGN}}}{4\nu}, \tag{13}$$

where

$$h_{\mathrm{RGN}} = 2\left(\sum_{a=1}^{n_{en}} |\mathbf{r} \cdot \nabla N_a| \right)^{-1}, \quad \mathbf{r} = \frac{\nabla \|\mathbf{u}^h\|}{\|\nabla \|\mathbf{u}^h\|\|}. \tag{14}$$

The parameter $\tau^{e}_{\mathrm{SUGN3}}$ is defined as

$$\tau^{e}_{\mathrm{SUGN3}} = \frac{(h^e_{\mathrm{RGN}})^2}{4\nu^e}, \tag{15}$$

where ν^e is the "kinematic viscosity" for the energy equation,

$$h^e_{\mathrm{RGN}} = 2\left(\sum_{a=1}^{n_{en}} |\mathbf{r}^e \cdot \nabla N_a| \right)^{-1}, \quad \mathbf{r}^e = \frac{\nabla \theta^h}{\|\nabla \theta^h\|} \tag{16}$$

and θ is the temperature. The parameters $(\tau^{\rho}_{\mathrm{SUPG}})_{\mathrm{UGN}}$, $(\tau^{u}_{\mathrm{SUPG}})_{\mathrm{UGN}}$ and $(\tau^{e}_{\mathrm{SUPG}})_{\mathrm{UGN}}$ are calculated from their components by using the "r-switch":

$$(\tau^{\rho}_{\mathrm{SUPG}})_{\mathrm{UGN}} = \left(\frac{1}{(\tau^{\rho}_{\mathrm{SUGN1}})^r} + \frac{1}{(\tau^{\rho}_{\mathrm{SUGN2}})^r} \right)^{-\frac{1}{r}}, \tag{17}$$

$$(\tau^{u}_{\mathrm{SUPG}})_{\mathrm{UGN}} = \left(\frac{1}{(\tau^{u}_{\mathrm{SUGN1}})^r} + \frac{1}{(\tau^{u}_{\mathrm{SUGN2}})^r} + \frac{1}{(\tau^{u}_{\mathrm{SUGN3}})^r} \right)^{-\frac{1}{r}}, \tag{18}$$

$$(\tau^{e}_{\mathrm{SUPG}})_{\mathrm{UGN}} = \left(\frac{1}{(\tau^{e}_{\mathrm{SUGN1}})^r} + \frac{1}{(\tau^{e}_{\mathrm{SUGN2}})^r} + \frac{1}{(\tau^{e}_{\mathrm{SUGN3}})^r} \right)^{-\frac{1}{r}}. \tag{19}$$

This "r-switch" was first introduced in [20]. Typically, $r = 2$.

As the first option in defining the shock-capturing term, first the "shock-capturing viscosity" ν_{SHOC} is defined:

$$\nu_{\mathrm{SHOC}} = \tau_{\mathrm{SHOC}}(u_{\mathrm{int}})^2, \tag{20}$$

where

$$\tau_{\mathrm{SHOC}} = \frac{h_{\mathrm{SHOC}}}{2u_{\mathrm{cha}}} \left(\frac{\|\nabla \rho^h\| h_{\mathrm{SHOC}}}{\rho_{\mathrm{ref}}} \right)^{\beta}, \tag{21}$$

$$h_{\mathrm{SHOC}} = h_{\mathrm{JGN}}, \tag{22}$$

$$h_{\mathrm{JGN}} = 2\left(\sum_{a=1}^{n_{en}} |\mathbf{j}\cdot\nabla N_a|\right)^{-1}. \tag{23}$$

Here ρ_{ref} is a reference density (such as ρ^h at the inflow, or the difference between the estimated maximum and minimum values of ρ^h), u_{cha} is a characteristic velocity (such as u_{ref} or $\|\mathbf{u}^h\|$ or acoustic speed c), and u_{int} is an intrinsic velocity (such as u_{cha} or $\|\mathbf{u}^h\|$or acoustic speed c). Typically, $u_{\mathrm{int}} = u_{\mathrm{cha}} = u_{\mathrm{ref}}$. The parameter β influences the smoothness of the shock-front. It is set as $\beta = 1$ for smoother shocks and $\beta = 2$ for sharper shocks (in return for tolerating possible overshoots and undershoots). The compromise between the $\beta = 1$ and 2 selections is defined as the following averaged expression for τ_{SHOC}:

$$\tau_{\mathrm{SHOC}} = \frac{1}{2}\left((\tau_{\mathrm{SHOC}})_{\beta=1} + (\tau_{\mathrm{SHOC}})_{\beta=2}\right). \tag{24}$$

As another option for calculating the shock-capturing parameter, ν_{SHOC} is defined as

$$\nu_{\mathrm{SHOC}} = \|\mathbf{Y}^{-1}\mathbf{Z}\|\left(\sum_{i=1}^{n_{\mathrm{sd}}}\left\|\mathbf{Y}^{-1}\frac{\partial\mathbf{U}^h}{\partial x_i}\right\|^2\right)^{\beta/2-1}\left(\frac{h_{\mathrm{SHOC}}}{2}\right)^{\beta}, \tag{25}$$

where $\mathbf{Y}$ is a diagonal scaling matrix constructed from the reference values of the components of $\mathbf{U}$:

$$\mathbf{Y} = \begin{bmatrix} (U_1)_{\mathrm{ref}} & 0 & 0 & 0 & 0 \\ 0 & (U_2)_{\mathrm{ref}} & 0 & 0 & 0 \\ 0 & 0 & (U_3)_{\mathrm{ref}} & 0 & 0 \\ 0 & 0 & 0 & (U_4)_{\mathrm{ref}} & 0 \\ 0 & 0 & 0 & 0 & (U_5)_{\mathrm{ref}} \end{bmatrix}, \tag{26}$$

$$\mathbf{Z} = \frac{\partial\mathbf{U}^h}{\partial t} + \mathbf{A}_i^h\frac{\partial\mathbf{U}^h}{\partial x_i} \tag{27}$$

or

$$\mathbf{Z} = \mathbf{A}_i^h\frac{\partial\mathbf{U}^h}{\partial x_i} \tag{28}$$

and $\beta = 1$ or $\beta = 2$. In a variation of the expression given by Eq. (25), ν_{SHOC} is defined by the following expression:

$$\nu_{\mathrm{SHOC}} = \|\mathbf{Y}^{-1}\mathbf{Z}\|\left(\sum_{i=1}^{n_{\mathrm{sd}}}\left\|\mathbf{Y}^{-1}\frac{\partial\mathbf{U}^h}{\partial x_i}\right\|^2\right)^{\beta/2-1}\|\mathbf{Y}^{-1}\mathbf{U}^h\|^{1-\beta}\left(\frac{h_{\mathrm{SHOC}}}{2}\right)^{\beta}. \tag{29}$$

The compromise between the $\beta = 1$ and 2 selections is defined as the following averaged expression for ν_{SHOC}:

$$\nu_{\mathrm{SHOC}} = \frac{1}{2}\left((\nu_{\mathrm{SHOC}})_{\beta=1} + (\nu_{\mathrm{SHOC}})_{\beta=2}\right). \tag{30}$$

Based on Eq. (25), a separate ν_{SHOC} can be calculated for each component of the test vector-function $\mathbf{W}$:

$$(\nu_{\mathrm{SHOC}})_{\mathrm{I}} = |(\mathbf{Y}^{-1}\mathbf{Z})_{\mathrm{I}}|\left(\sum_{i=1}^{n_{\mathrm{sd}}}\left|\left(\mathbf{Y}^{-1}\frac{\partial\mathbf{U}^h}{\partial x_i}\right)_{\mathrm{I}}\right|^2\right)^{\beta/2-1}\left(\frac{h_{\mathrm{SHOC}}}{2}\right)^{\beta}, \quad I = 1, 2, \ldots, n_{\mathrm{sd}} + 2. \tag{31}$$

Similarly, a separate ν_{SHOC} for each component of $\mathbf{W}$ can be calculated based on Eq. (29):

$$(\nu_{\mathrm{SHOC}})_{\mathrm{I}} = |(\mathbf{Y}^{-1}\mathbf{Z})_{\mathrm{I}}| \left(\sum_{i=1}^{n_{\mathrm{sd}}} \left| \left(\mathbf{Y}^{-1} \frac{\partial \mathbf{U}^h}{\partial x_i} \right)_{\mathrm{I}} \right|^2 \right)^{\beta/2-1} |(\mathbf{Y}^{-1}\mathbf{U}^h)_{\mathrm{I}}|^{1-\beta} \left(\frac{h_{\mathrm{SHOC}}}{2} \right)^{\beta}, \quad I = 1, 2, \ldots, n_{\mathrm{sd}} + 2. \tag{32}$$

Given ν_{SHOC}, the shock-capturing term is defined as

$$S_{\mathrm{SHOC}} = \sum_{e=1}^{n_{\mathrm{el}}} \int_{\Omega^e} \nabla \mathbf{W}^h : (\boldsymbol{\kappa}_{\mathrm{SHOC}} \cdot \nabla \mathbf{U}^h) \, \mathrm{d}\Omega, \tag{33}$$

where $\boldsymbol{\kappa}_{\mathrm{SHOC}}$ is defined as $\boldsymbol{\kappa}_{\mathrm{SHOC}} = \nu_{\mathrm{SHOC}}\mathbf{I}$. As a possible alternative, it is defined as $\boldsymbol{\kappa}_{\mathrm{SHOC}} = \nu_{\mathrm{SHOC}}\,\mathbf{jj}$. If the option given by Eq. (31) or Eq. (32) is exercised, then ν_{SHOC} becomes an $(n_{\mathrm{sd}} + 2) \times (n_{\mathrm{sd}} + 2)$ diagonal matrix, and the matrix $\boldsymbol{\kappa}_{\mathrm{SHOC}}$ becomes augmented from an $n_{\mathrm{sd}} \times n_{\mathrm{sd}}$ matrix to an $(n_{\mathrm{sd}} \times (n_{\mathrm{sd}} + 2)) \times ((n_{\mathrm{sd}} + 2) \times n_{\mathrm{sd}})$ matrix.

To preclude compounding, ν_{SHOC} can be modified as follows:

$$\nu_{\mathrm{SHOC}} \leftarrow \nu_{\mathrm{SHOC}} - \mathrm{switch}(\tau_{\mathrm{SUPG}}(\mathbf{j} \cdot \mathbf{u})^2,\ \tau_{\mathrm{SUPG}}(|\mathbf{j} \cdot \mathbf{u}| - c)^2,\ \nu_{\mathrm{SHOC}}), \tag{34}$$

where the "switch" function is defined as the "min" function or as the "r-switch" used earlier in this section. For viscous flows, the above modification would be made separately with each of $\tau^{\rho}_{\mathrm{SUPG}}$, τ^{u}_{SUPG} and τ^{e}_{SUPG}, and this would result in ν_{SHOC} becoming a diagonal matrix even if the option given by Eq. (31) or Eq. (32) is not exercised.

5. Test computations

The test computations were carried out by using the space–time SUPG formulation described in Section 3.2. We used two steady-state, inviscid test problems: "oblique shock" and "reflected shock". These were

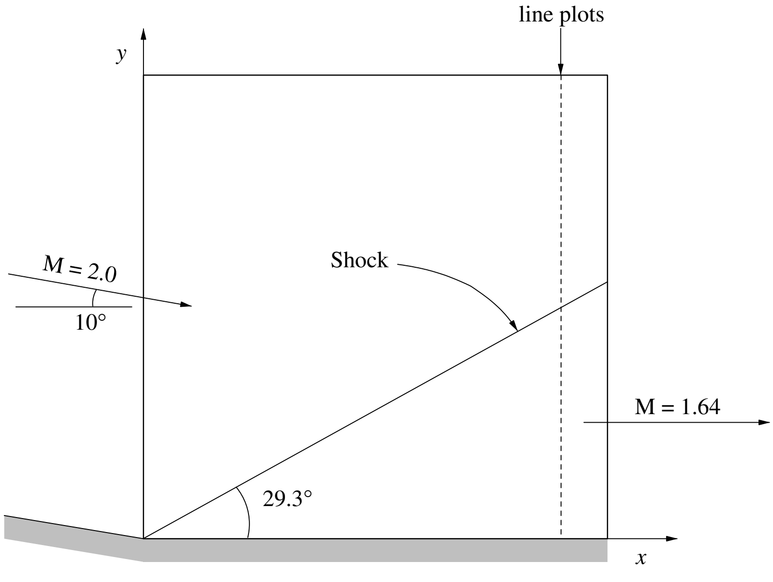

Fig. 1. Oblique shock. Problem description.

used in many earlier publications, and here we compute each of them with two different options for the shock-capturing parameter. In the option denoted by "CYZ12", we use Eq. (25) with Eq. (30), and in the option denoted by "CYZU12", Eq. (29) with Eq. (30). In both options, we use for $\mathbf{Z}$ the expression given by Eq. (28), and set $\boldsymbol{\kappa}_{\mathrm{SHOC}} = \nu_{\mathrm{SHOC}}\mathbf{I}$. With both options, as stabilization parameters, we use Eq. (11), and in Eqs. (17)–(19) we do not include τ_{SUGN2}. In both problems, the time-step size is 0.05. The num-

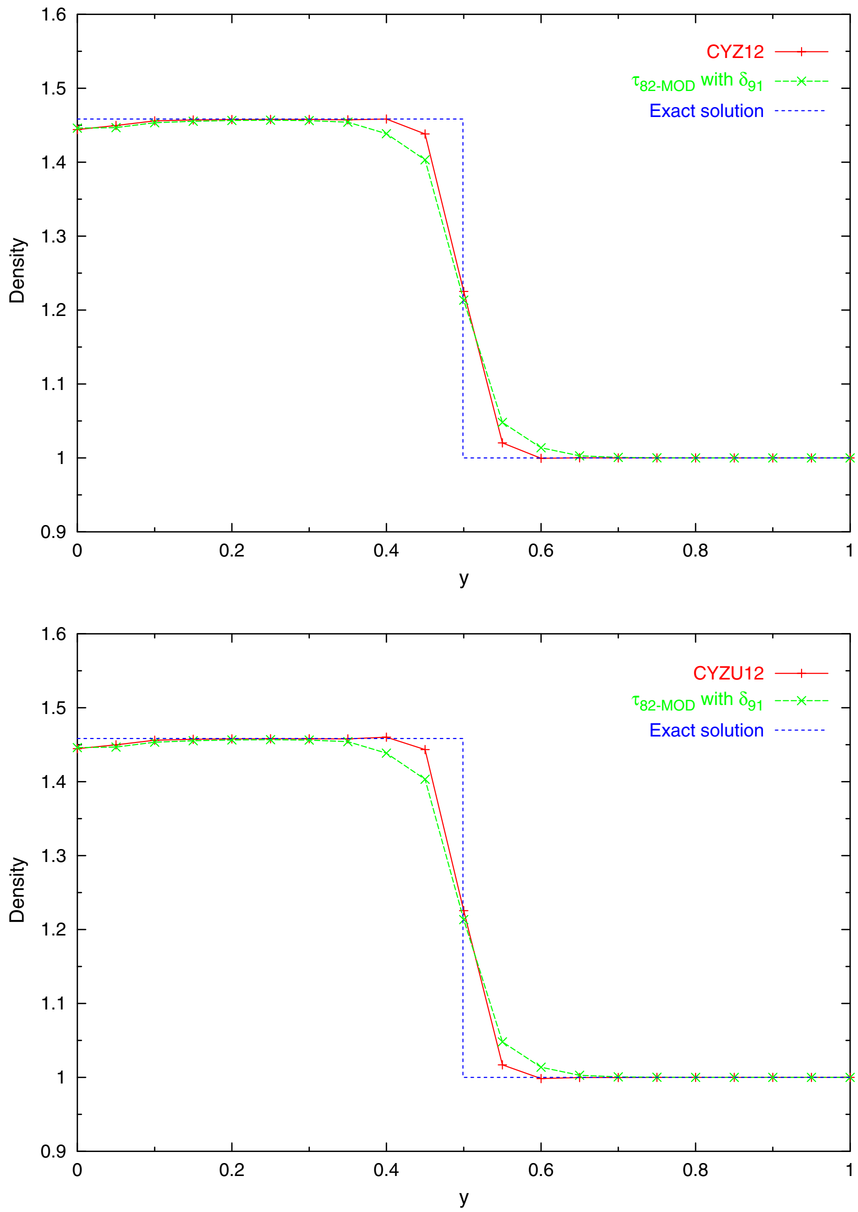

Fig. 2. Oblique shock. Density along $x = 0.9$, obtained with CYZ12 (top) and CYZU12 (bottom), compared with the solution obtained with the $\tau_{82\text{-MOD}}$ and δ_{91} combination.

ber of time steps, nonlinear iterations, and the inner and outer GMRES iterations are 100, 3, 30, and 2, respectively. The results are compared to those obtained with the $\tau_{82\text{-MOD}}$ and δ_{91} combination. The version of $\tau_{82\text{-MOD}}$ used in this paper for comparison is similar to the one given in [21]:

$$\tau_{82\text{-MOD}} = \max(0, \zeta(\tau_a - \tau_\delta)), \tag{35}$$

where

$$\tau_a = \frac{h_{\mathrm{BGN}}}{2u_{cc}}, \quad \tau_\delta = \frac{\delta_{91}}{(u_{cc})^2}, \quad u_{cc} = c + \left| \mathbf{u}^h \cdot \frac{\boldsymbol{\nabla}\|\mathbf{U}^h\|}{\|\boldsymbol{\nabla}\|\mathbf{U}^h\|\|} \right|, \tag{36}$$

$$\zeta = \frac{u_{cc}\Delta t/h_{\mathrm{BGN}}}{1 + u_{cc}\Delta t/h_{\mathrm{BGN}}}, \quad h_{\mathrm{BGN}} = 2\left(\sum_{a=1}^{n_{en}} \left| \frac{\boldsymbol{\nabla}\|\mathbf{U}^h\|}{\|\boldsymbol{\nabla}\|\mathbf{U}^h\|\|} \cdot \boldsymbol{\nabla} N_a \right| \right)^{-1}. \tag{37}$$

Oblique shock. Fig. 1 shows the problem description. This is a Mach 2 uniform flow over a wedge at an angle of $-10°$ with the horizontal wall. The solution involves an oblique shock at an angle of 29.3° emanating from the leading edge. The computational domain is a square with $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. The inflow conditions are given as $M = 2.0$, $\rho = 1.0$, $u_1 = \cos 10°$, $u_2 = -\sin 10°$, and $p = 0.179$. This results in an exact solution with the following outflow data: $M = 1.64$, $\rho = 1.46$, $u_1 = 0.887$, $u_2 = 0.0$, and $p = 0.305$. All essential boundary conditions are imposed at the left and top boundaries, slip condition at the wall, and no boundary conditions at the right boundary. The mesh is uniform and consists of 20×20 elements. Fig. 2 shows the density along $x = 0.9$, obtained with CYZ12 and CYZU12, compared with the solution obtained with the $\tau_{82\text{-MOD}}$ and δ_{91} combination. In addition to being much simpler, the new shock-capturing parameters yield shocks with better quality.

Reflected shock. Fig. 3 shows the problem description. This problem consists of three flow regions (R1, R2 and R3) separated by an oblique shock and its reflection from the wall. The computational domain is a rectangle with $0 \leqslant x \leqslant 4.1$ and $0 \leqslant y \leqslant 1$. The inflow conditions in R1 are given as $M = 2.9$, $\rho = 1.0$, $u_1 = 2.9$, $u_2 = 0.0$, and $p = 0.7143$. Specifying these conditions and requiring the incident shock to be at an angle of 29° results in an exact solution with the following flow data: R2: $M = 2.378$, $\rho = 1.7$, $u_1 = 2.619$, $u_2 = -0.506$, and $p = 1.528$; R3: $M = 1.942$, $\rho = 2.687$, $u_1 = 2.401$, $u_2 = 0.0$, and $p = 2.934$. All essential boundary conditions are imposed at the left and top boundaries, slip condition at the wall, and no boundary conditions at the right boundary. The mesh is uniform and consists of 60×20 elements. Fig. 4 shows the density along $y = 0.25$, obtained with CYZ12 and CYZU12, compared with the solution obtained with the $\tau_{82\text{-MOD}}$ and δ_{91} combination. Again, the new, much simpler shock-capturing parameters yield shocks with better quality.

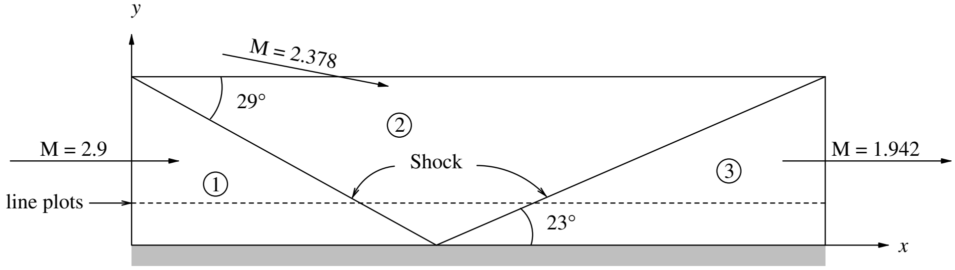

Fig. 3. Reflected shock. Problem description.

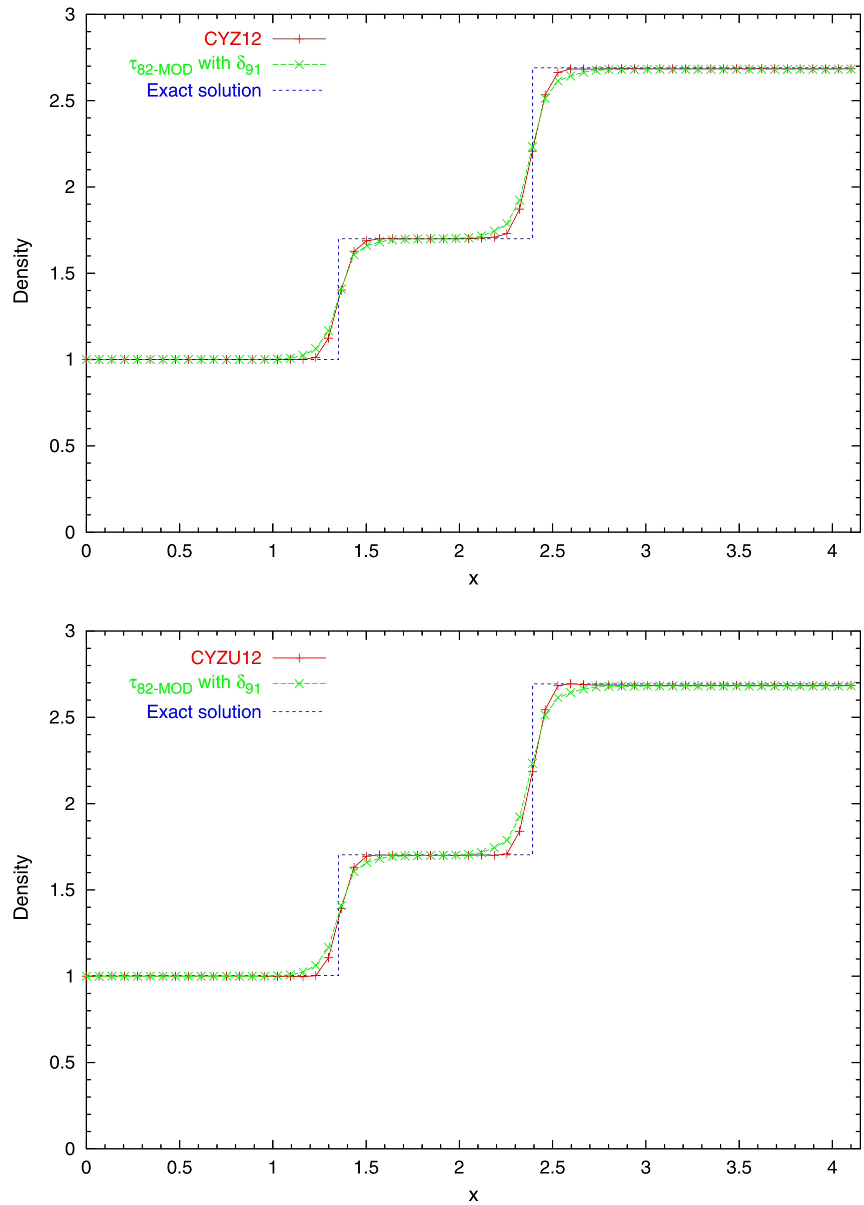

Fig. 4. Reflected shock. Density along $y = 0.25$, obtained with CYZ12 (top) and CYZU12 (bottom), compared with the solution obtained with the $\tau_{82\text{-MOD}}$ and δ_{91} combination.

6. Concluding remarks

We described, for the streamline-upwind/Petrov–Galerkin (SUPG) formulation of compressible flows based on conservation variables, new ways for determining the stabilization and shock-capturing parameters. The stabilization parameter, which is typically known as "τ", plays an important role in determining the accuracy of the solutions. The shock-capturing term provides additional stabilization near the shocks, and how the shock-capturing parameter it involves is defined influences the quality of the solution near the shocks. These new ways of calculating the τs and shock-capturing parameters are partly based on the ideas underlying the τs and and DCDD stabilization developed for incompressible flows. Compared to the earlier shock-capturing parameter that was derived based on the entropy variables, the new ones are much simpler, involve less operations in calculating the shock-capturing term, and gave better shock resolution in the test computations we carried out.

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