Computer Methods in Applied Mechanics and Engineering 104 (1993) 31–48
North-Holland

CMA 330

Stabilized finite element methods for the velocity–pressure–stress formulation of incompressible flows

Marek A. Behr
Department of Aerospace Engineering and Mechanics, Army High Performance Computing Research Center and Minnesota Supercomputer Institute, 1200 Washington Avenue South, Minneapolis, MN 55415, USA

Leopoldo P. Franca
Laboratório Nacional de Computação Científica (LNCC/CNPq), Rua Lauro Müller 455, 22290 Rio de Janeiro, RJ, Brazil

Tayfun E. Tezduyar
Department of Aerospace Engineering and Mechanics, Army High Performance Computing Research Center and Minnesota Supercomputer Institute 1200 Washington Avenue South, Minneapolis, MN 55415, USA

Received 31 January 1992

Formulated in terms of velocity, pressure and the extra stress tensor, the incompressible Navier–Stokes equations are discretized by stabilized finite element methods. The stabilized methods proposed are analyzed for a linear model and extended to the Navier–Stokes equations. The numerical tests performed confirm the good stability characteristics of the methods. These methods are applicable to various combinations of interpolation functions, including the simplest equal-order linear and bilinear elements.

1. Introduction

Aiming to develop stabilized finite element methods for viscoelastic flows, as a first step, we treat Newtonian flows formulated in terms of velocity, pressure and the extra stress tensor. In the recent finite element literature, one can find a number of methods proposed to solve the equations of viscoelastic flows in terms of these three variables (cf. [1–7]). In particular, continuous approximations for all variables are used by Marchal and Crochet [7] and are justified for the Stokes operator part of the equations in [6]. To achieve a stable approximation, a rather complex combination of interpolations is used in [7]; the stresses and the velocity are in some sense higher-order with respect to the pressure interpolation.

Correspondence to: Dr. Leopoldo P. Franca, Laboratório Nacional de Computação Científica (LNCC/CNPq), Rua Lauro Müller 445, 2290 Rio de Janeiro, RJ, Brazil.

Recently, Franca and Stenberg [8] proposed a velocity–pressure–extra stress formulation for the Stokes flow. This formulation is based on a stabilized finite element method that allows the use of rather simple combinations of interpolations, including equal-order linear and bilinear elements. In this paper, we extend this formulation to include the inertia terms in the momentum equations. The parameters in the added stabilizing terms are designed to take into account locally advective and locally diffusive dominated elements (cf. [9–13] and references therein). Two versions of the stabilized method are proposed. In Section 3, we perform a convergence analysis for the stabilized method based on a linearized model of incompressible flows. As far as the authors are aware, the previous work that included convergence analysis was limited to the Stokes operator. Aside from a recent report [3] considering nonzero Weissenberg numbers, our work is one of the first attempts to analyze these equations at nonzero Reynolds numbers (i.e., in the presence of both the Stokes and the advection operators). It should be noted that the major concern in computation of viscoelastic flows heretofore has been related to the advection of the extra stress tensor (i.e., the cases with nonzero Weissenberg numbers). Here we restrict ourselves to dealing with the advection terms in the momentum equations (i.e., the cases with positive Reynolds numbers).

In Section 4 we extend the formulation to the Navier–Stokes equations, and in Section 5 we report the test results obtained with equal-order bilinear elements. The tests reported are restricted to steady flows.

2. Stabilized methods for the velocity–pressure–stress formulation

Written in terms of velocity, pressure and the extra stress, the steady-state, linearized, incompressible Navier–Stokes model is given as

$$\begin{aligned} &\frac{1}{2\nu}\boldsymbol{T}-\boldsymbol{\varepsilon}(\boldsymbol{u})=\boldsymbol{0} \quad \text{in } \Omega\,, \qquad \nabla\cdot\boldsymbol{u}=0 \quad \text{in } \Omega\,, \\ &(\nabla\boldsymbol{u})\boldsymbol{a}-\nabla\cdot\boldsymbol{T}+\nabla p=\boldsymbol{f} \quad \text{in } \Omega\,, \qquad \boldsymbol{u}=\boldsymbol{0} \quad \text{on } \Gamma\,, \end{aligned} \tag{1}$$

where $\boldsymbol{T}$ is the extra stress tensor, $\boldsymbol{u}$ is the velocity, p is the pressure, ν is the viscosity. $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the symmetric part of the velocity gradient, $\boldsymbol{a}$ is a given velocity field, and $\boldsymbol{f}$ is the body force. In our notation, both $\boldsymbol{T}$ and p are scaled with the density. The model is formulated on a bounded domain $\Omega\subset\mathbb{R}^N$, $N=2,3$, with a polyhedral boundary Γ. We consider homogeneous Dirichlet boundary conditions to simplify the arguments in the analysis that follows. More general boundary conditions are used in the nonlinear model in Section 4 and in the numerical results.

Next, a partition $\mathscr{C}_h$ of $\bar{\Omega}$ into elements consisting of triangles (tetrahedrons in $\mathbb{R}^3$) or convex quadrilaterals (hexahedrons in $\mathbb{R}^3$) is performed in the usual way (i.e., no overlapping is allowed between any two elements of the partition; the union of all element domains K reproduces $\bar{\Omega}$, etc.), and combinations of triangles and quadrilaterals for the two-dimensional cases can be accommodated. For convenience, we adopt the following notation:

$$R_m(K)=\begin{cases} P_m(K)\,, & \text{if } K \text{ is a triangle or tetrahedron}\,,\\ Q_m(K)\,, & \text{if } K \text{ is a quadrilateral or hexahedron}\,,\end{cases}$$

where for each integer $m \geq 0$, P_m and Q_m are the spaces of all polynomials of degree $\leq m$ in the variables $x_1, x_2, \ldots, x_N$ – P_m with respect to all combinations of these variables and Q_m with respect to each one of them.

The finite element spaces considered are standard, and are restricted to *any* combination of continuous interpolations given by

$$V_h = \{v \in (H_0^1(\Omega))^N \mid v|_K \in (R_k(K))^N,\ K \in \mathscr{C}_h\}\,, \tag{2}$$

$$P_h = \{q \in \mathscr{C}^0(\Omega) \cap L_0^2(\Omega) \mid q|_K \in R_l(K),\ K \in \mathscr{C}_h\}\,, \tag{3}$$

$$W_h = \{S \in (\mathscr{C}^0(\Omega))^{N^2} \mid S|_K \in (R_m(K))^{N^2},\ K \in \mathscr{C}_h\}\,, \tag{4}$$

where the integers k, l and m denote the order of the finite element polynomial approximations for velocity, pressure and the extra stress, respectively. *Any* combination of k, l and m might be used in the methods that follow, a possibility which is not accommodated in the standard Galerkin method, in general. For the notation in (2)–(4), as usual, $L^2(\Omega)$ is the space of square-integrable functions in Ω; $L_0^2(\Omega)$, the space of L^2-functions with zero mean value in Ω; $\mathscr{C}^0(\Omega)$, the space of continuous functions in Ω; and $H_0^1(\Omega)$ is the Sobolev space of functions with square-integrable value and derivatives in Ω with zero value on the boundary Γ. We employ $(\cdot\,,\,\cdot)$ to denote the L^2-inner product in Ω and $\|\cdot\|_0$, the $L^2(\Omega)$-norm. Also, $(\cdot\,,\,\cdot)_K$ and $\|\cdot\|_{0,K}$ are used to denote the L^2-inner product and norm in the element domain K, respectively; and, the H^1-norm is denoted by $\|\cdot\|_1$.

Our first method (Method I) may be viewed as an extension of the method proposed in [8] to the Stokes flow in terms of velocity, pressure and the extra stress, and can be written as follows: Find $u_h \in V_h$, $p_h \in P_h$ and $T_h \in W_h$ such that

$$B_1(T_h, p_h, u_h; S, q, v) = F_1(S, q, v)\,, \quad (S, q, v) \in W_h \times P_h \times V_h\,, \tag{5}$$

with

$$\begin{aligned} B_1(T, p, u; S, q, v) = {} & \frac{1}{2\nu}(T, S) - (\varepsilon(u), S) + (\nabla\cdot u, q) - (T, \varepsilon(v)) - ((\nabla u)a, v) \\ & + (p, \nabla\cdot v) - (\nabla\cdot u, \delta\nabla\cdot v) - \alpha 2\nu\Big(\frac{1}{2\nu}T - \varepsilon(u), \frac{1}{2\nu}S - \varepsilon(v)\Big) \\ & + \sum_{K\in\mathscr{C}_h}((\nabla u)a + \nabla p - \nabla\cdot T, \tau(-(\nabla v)a + \nabla q - \nabla\cdot S))_K \end{aligned} \tag{6}$$

and

$$F_1(S, q, v) = -(f, v) + \sum_{K\in\mathscr{C}_h}(f, \tau(-(\nabla v)a + \nabla q - \nabla\cdot S))_K\,,$$

where the stability parameters δ, τ and α are defined as

$$\delta = \lambda|a(x)|_p h_K \xi(\mathrm{Re}_K(x))\,, \tag{8}$$

$$\tau(x, \mathrm{Re}_K(x)) = \frac{h_K}{2|a(x)|_p}\, \xi(\mathrm{Re}_K(x)) , \tag{9}$$

$$0 < \alpha \leq 0.5 , \tag{10}$$

$$\mathrm{Re}_K(x) = \frac{m_m |a(x)|_p h_K}{4\nu(x)} , \tag{11}$$

$$\xi(\mathrm{Re}_K(x)) = \begin{cases} \mathrm{Re}_K(x) , & 0 \leq \mathrm{Re}_K(x) < 1 , \\ 1 , & \mathrm{Re}_K(x) \geq 1 , \end{cases} \tag{12}$$

$$|a(x)|_p = \begin{cases} \left(\sum_{i=1}^{N} |a_i(x)|^p \right)^{1/p} , & 1 \leq p < \infty , \\ \max_{i=1,N} |a_i(x)| , & p = \infty , \end{cases} \tag{13}$$

$$m_m = \min\{ \tfrac{1}{3}, 2C_m \} , \tag{14}$$

$$C_m \sum_{K \in \mathscr{C}_h} h_K^2 \|\nabla \cdot T\|_{0,K}^2 \leq \|T\|_0^2 , \quad T \in W_h , \tag{15}$$

and λ is a positive parameter.

REMARK 1. This formulation is an extension of the one proposed in [8] for Stokes formulation to include the advective terms, and the design of the stabilization parameters τ and δ closely follows the definitions given in [10] which coincides with the design of τ in [13] for low-order elements.

REMARK 2. Unlike the stabilized u–p formulation (thoroughly studied for equal-order linear and bilinear elements by Tezduyar et al. [11–13] and refs. therein), the present formulation naturally accounts for the Laplacian of the velocity in the momentum equation, as part of the additional τ stabilizing terms. Here, taken into account by $\nabla \cdot T$, this term does not vanish for linear or bilinear approximations of the extra stress tensor T, whereas in the u–p formulation, the Laplacian of the velocity is identically zero for linear approximations. In some sense, the present formulation is more consistent for low-order approximations, since all terms are present in the formulation.

REMARK 3. It is possible to employ *discontinuous* extra stresses with order $m \geq k - 1$ for triangles and $m \geq k$ for quadrilaterals. In this case the inf–sup condition

$$\sup_{0 \neq T \in W_h^d} \frac{(T, \varepsilon(u))}{\|T\|_0} \geq C\|u\|_1 , \quad u \in V_h \tag{16}$$

is automatically satisfied, and we have a stable method without the need for the α-term. Furthermore, in this case, the extra stress can be eliminated at the element level and we are left with a different u–p formulation as compared to the stabilized methods proposed directly in terms of velocity and pressure [10–13]. It is intriguing that so far we have been able to obtain good numerical results with equal-order bilinear elements, without the need to employ the α-term, *even for continuous extra stress interpolation.*

Before continuing further, let us note that if we replace $\boldsymbol{v}$ by $-\boldsymbol{v}$ in the definition of Method I, then this method can also be expressed as follows: Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$, $p_h \in P_h$ and $\boldsymbol{T}_h \in \boldsymbol{W}_h$ such that

$$B_{\mathrm{I}}(\boldsymbol{T}_h, p_h, \boldsymbol{u}_h; \boldsymbol{S}, q, \boldsymbol{v}) = F_{\mathrm{I}}(\boldsymbol{S}, q, \boldsymbol{v}), \quad (\boldsymbol{S}, q, \boldsymbol{v}) \in \boldsymbol{W}_h \times P_h \times \boldsymbol{V}_h, \tag{17}$$

with

$$\begin{aligned} B_{\mathrm{I}}(\boldsymbol{T}, p, \boldsymbol{u}; \boldsymbol{S}, q, \boldsymbol{v}) &= \frac{1}{2\nu}(\boldsymbol{T}, \boldsymbol{S}) - (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{S}) + (\nabla\cdot \boldsymbol{u}, q) + (\boldsymbol{T}, \boldsymbol{\varepsilon}(\boldsymbol{v})) + ((\nabla \boldsymbol{u})\boldsymbol{a}, \boldsymbol{v}) \\ &\quad -(p, \nabla\cdot\boldsymbol{v}) + (\nabla\cdot\boldsymbol{u}, \delta\nabla\cdot\boldsymbol{v}) + \alpha 2\nu\Big(\frac{1}{2\nu}\boldsymbol{T} - \boldsymbol{\varepsilon}(\boldsymbol{u}), -\frac{1}{2\nu}\boldsymbol{S} - \boldsymbol{\varepsilon}(\boldsymbol{v})\Big) \\ &\quad + \sum_{K\in\mathscr{C}_h} ((\nabla\boldsymbol{u})\boldsymbol{a} + \nabla p - \nabla\cdot\boldsymbol{T}, \tau((\nabla\boldsymbol{v})\boldsymbol{a} + \nabla q - \nabla\cdot\boldsymbol{S}))_K \end{aligned} \tag{18}$$

and

$$F_{\mathrm{I}}(\boldsymbol{S}, q, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + \sum_{K\in\mathscr{C}_h} (\boldsymbol{f}, \tau((\nabla\boldsymbol{v})\boldsymbol{a} + \nabla q - \nabla\cdot\boldsymbol{S}))_K. \tag{19}$$

In the arguments of B_{I} and F_{I} above, we relabeled $-\boldsymbol{v}$ by $\boldsymbol{v}$ again.

Note that the method may be implemented in any of the forms above, in principle, but the algebraic equations depend on the choice of these signs. We do not deal further with this aspect here. Instead, we wish to motivate the introduction of a variation of Method I. First we note that all additional terms are added in a least-squares fashion, except for the α-term. We will see in the next section that having the α-term with the current choices of signs restricts the possible values of α in the numerical simulation. A simple cure is to consider the following method (Method II): Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$, $p_h \in P_h$ and $\boldsymbol{T}_h \in \boldsymbol{W}_h$ such that

$$B_{\mathrm{II}}(\boldsymbol{T}_h, p_h, \boldsymbol{u}_h; \boldsymbol{S}, q, \boldsymbol{v}) = F_{\mathrm{II}}(\boldsymbol{S}, q, \boldsymbol{v}), \quad (\boldsymbol{S}, q, \boldsymbol{v}) \in \boldsymbol{W}_h \times P_h \times \boldsymbol{V}_h, \tag{20}$$

with

$$\begin{aligned} B_{\mathrm{II}}(\boldsymbol{T}, p, \boldsymbol{u}; \boldsymbol{S}, q, \boldsymbol{v}) &= \frac{1}{2\nu}(\boldsymbol{T}, \boldsymbol{S}) - (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{S}) + (\nabla\cdot \boldsymbol{u}, q) + (\boldsymbol{T}, \boldsymbol{\varepsilon}(\boldsymbol{v})) + ((\nabla \boldsymbol{u})\boldsymbol{a}, \boldsymbol{v}) \\ &\quad -(p, \nabla\cdot\boldsymbol{v}) + (\nabla\cdot\boldsymbol{u}, \delta\nabla\cdot\boldsymbol{v}) + \alpha 2\nu\Big(\frac{1}{2\nu}\boldsymbol{T} - \boldsymbol{\varepsilon}(\boldsymbol{u}), \frac{1}{2\nu}\boldsymbol{S} - \boldsymbol{\varepsilon}(\boldsymbol{v})\Big) \\ &\quad + \sum_{K\in\mathscr{C}_h} ((\nabla\boldsymbol{u})\boldsymbol{a} + \nabla p - \nabla\cdot\boldsymbol{T}, \tau((\nabla\boldsymbol{v})\boldsymbol{a} + \nabla q - \nabla\cdot\boldsymbol{S}))_K \end{aligned} \tag{21}$$

$$F_{\mathrm{II}}(\boldsymbol{S}, q, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + \sum_{K\in\mathscr{C}_h} (\boldsymbol{f}, \tau((\nabla\boldsymbol{v})\boldsymbol{a} + \nabla q - \nabla\cdot\boldsymbol{S}))_K. \tag{22}$$

The parameters are designed in the same way as in (8)–(15), except that now α does not need to have an upper bound as indicated in (10). For Method II, α may be chosen arbitrarily greater than zero.

3. Error analysis

In this section, we study the convergence of the stabilized Method I given by (5)–(15). We prove the following result.

THEOREM 3.1. Assuming the given data $a(x)$ and $\nu(x)$ to satisfy
(i) $\nabla \cdot a(x) = 0$,
(ii) $\nu(x) = \nu = \text{constant} > 0$;
if, in addition, the solution to (1) satisfies $u \in H^{k+1}(\Omega^N) \cap H_0^1(\Omega)^N$, $p \in H^{l+1}(\Omega) \cap L_0^2(\Omega)$ and $T \in H^{m+1}(\Omega)^{N(N+1)/2}$, then the solution (T_h, p_h, u_h) of the method given by (5)–(15) converges to (T, p, u), the solution of (1), as follows:

$$\begin{aligned}
&\frac{1}{8\nu}\|T_h - T\|_0^2 + 2\nu\alpha\|\varepsilon(u_h - u)\|_0^2 + \frac{\alpha}{1+\alpha}\|\tau^{1/2}(\nabla(u_h - u)a + \nabla(p_h - p))\|_0^2 \\
&\quad + \|\delta^{1/2}\nabla\cdot(u_h - u)\|_0^2 \le C(\alpha)\sum_{K\in\mathscr{C}_h}((2\nu)^{-1}h_K^{2m+2}|T|_{m+1,K}^2 \\
&\quad + h_K^{2l}|p|_{l+1,K}^2(H(\mathrm{Re}_K - 1)h_K\sup_{x\in K}|a|_p^{-1} + H(1-\mathrm{Re}_K)h_K^2(2\nu)^{-1}) \\
&\quad + h_K^{2k}|u|_{k+1,K}^2(H(\mathrm{Re}_K - 1)h_K\sup_{x\in K}|a|_p + H(1-\mathrm{Re}_K)2\nu))\,,
\end{aligned}$$

where $H(\cdot)$ is the Heaviside function given by

$$H(x-y) = \begin{cases} 0\,, & x<y\,,\\ 1\,, & x>y\,. \end{cases} \tag{23}$$

Before establishing this theorem let us verify a few preliminary results. First note that, by definition, the stability parameter τ is bounded by a constant in each element domain K. In fact,

$$\tau(x, \mathrm{Re}_K(x)) = \frac{h_K}{2|a(x)|_p}\,, \quad \mathrm{Re}_K(x) \ge 1\,, \tag{24a}$$

$$\tau(x, \mathrm{Re}_K(x)) = \frac{m_m h_K^2}{8\nu}\,, \quad 0 \le \mathrm{Re}_K(x) < 1\,, \tag{24b}$$

and therefore, for $\mathrm{Re}_K(x) \ge 1$,

$$\tau(x, \mathrm{Re}_K(x)) = \frac{h_K}{2|a(x)|_p}\,\frac{1}{\mathrm{Re}_K(x)}\,\frac{m_m|a(x)|_p h_K}{4\nu} \le \frac{m_m h_K^2}{8\nu} \tag{25}$$

and combining with the definition (24b), we conclude that the bound (25) is valid for all values of $\mathrm{Re}_K(x)$.

The stability of the method can now be established.

LEMMA 3.1. (Stability). Under the same assumptions (i) and (ii) of Theorem 3.1, we have *for all* $(\boldsymbol{T}, p, \boldsymbol{u}) \in \boldsymbol{W}_h \times P_h \times \boldsymbol{V}_h$,

$$B_1(\boldsymbol{T}, p, \boldsymbol{u}; \boldsymbol{T}, p, -\boldsymbol{u}) \geqslant \frac{1}{8\nu} \|\boldsymbol{T}\|_0^2 + 2\nu\alpha \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0^2 + \frac{\alpha}{1+\alpha} \|\tau^{1/2}((\nabla \boldsymbol{u})\boldsymbol{a} + \nabla p)\|_0^2 + \|\delta^{1/2}\nabla \cdot \boldsymbol{u}\|_0^2 .$$

PROOF. By assumption (i), by (2) and integrating by parts, we have

$$((\nabla \boldsymbol{u})\boldsymbol{a}, \boldsymbol{u}) = 0 , \quad \boldsymbol{u} \in \boldsymbol{V}_h .$$

Let $\gamma > 1$ be a parameter to be set below. Then

$$\begin{aligned}
&B_1(\boldsymbol{T}, p, \boldsymbol{u}; \boldsymbol{T}, p, -\boldsymbol{u}) \\
&= \frac{1}{2\nu} \|\boldsymbol{T}\|_0^2 + 0 + \|\delta^{1/2}\nabla \cdot \boldsymbol{u}\|_0^2 + 2\nu\alpha \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0^2 - \frac{\alpha}{2\nu} \|\boldsymbol{T}\|_0^2 \\
&\quad + \sum_{K \in \mathscr{C}_h} \|\tau^{1/2}((\nabla \boldsymbol{u})\boldsymbol{a} + \nabla p - \nabla \cdot \boldsymbol{T})\|_{0,K}^2 \\
&= \frac{1-\alpha}{2\nu} \|\boldsymbol{T}\|_0^2 + 2\nu\alpha \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0^2 + \|\delta^{1/2}\nabla \cdot \boldsymbol{u}\|_0^2 \\
&\quad + \|\tau^{1/2}((\nabla \boldsymbol{u})\boldsymbol{a} + \nabla p)\|_0^2 + \sum_{K \in \mathscr{C}_h} \|\tau^{1/2}\nabla \cdot \boldsymbol{T}\|_{0,K}^2 - 2 \sum_{K \in \mathscr{C}_h} (\nabla \cdot \boldsymbol{T}, \tau((\nabla \boldsymbol{u})\boldsymbol{a} + \nabla p))_K \\
&\geqslant \frac{1-\alpha}{2\nu} \|\boldsymbol{T}\|_0^2 + 2\nu\alpha \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0^2 + \|\delta^{1/2}\nabla \cdot \boldsymbol{u}\|_0^2 \\
&\quad + (1-\gamma^{-1}) \|\tau^{1/2}((\nabla \boldsymbol{u})\boldsymbol{a} + \nabla p)\|_0^2 + (1-\gamma) \sum_{K \in \mathscr{C}_h} \|\tau^{1/2}\nabla \cdot \boldsymbol{T}\|_{0,K}^2 \\
&\geqslant \frac{1}{4\nu} (2 - 2\alpha + 1 - \gamma) \|\boldsymbol{T}\|_0^2 + 2\nu\alpha \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0^2 + \|\delta^{1/2}\nabla \cdot \boldsymbol{u}\|_0^2 \\
&\quad + (1-\gamma^{-1}) \|\tau^{1/2}((\nabla \boldsymbol{u})\boldsymbol{a} + \nabla p)\|_0^2 \\
&\geqslant \frac{2-3\alpha}{4\nu} \|\boldsymbol{T}\|_0^2 + 2\nu\alpha \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0^2 + \|\delta^{1/2}\nabla \cdot \boldsymbol{u}\|_0^2 + \frac{\alpha}{1+\alpha} \|\tau^{1/2}((\nabla \boldsymbol{u})\boldsymbol{a} + \nabla p)\|_0^2 \\
&\geqslant \frac{1}{8\nu} \|\boldsymbol{T}\|_0^2 + 2\nu\alpha \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0^2 + \|\delta^{1/2}\nabla \cdot \boldsymbol{u}\|_0^2 + \frac{\alpha}{1+\alpha} \|\tau^{1/2}((\nabla \boldsymbol{u})\boldsymbol{a} + \nabla p)\|_0^2 . \qquad \square
\end{aligned} \tag{26}$$

The second last line was obtained by setting $\gamma = 1 + \alpha > 1$, and recalling that α is to be selected between 0 and $\frac{1}{2}$, the result follows in the last line. In addition to the stability result above, we will need the following *interpolation estimate*.

LEMMA 3.2. Denoting by $\boldsymbol{\eta}_T = \tilde{\boldsymbol{T}}_h - \boldsymbol{T}$, $\eta_p = \tilde{p}_h - p$ *and* $\boldsymbol{\eta}_u = \tilde{\boldsymbol{u}}_h - \boldsymbol{u}$ *the interpolation errors and assuming the solution to (1) to be smooth enough as stated in Theorem 3.1, then for each* $K \in \mathscr{C}_h$, *we have*

(a) *If* $\mathrm{Re}_K \geq 1$, $\forall x \in K$, *then*

$$\frac{1}{2\nu}\|\eta_T\|^2_{0,K} + \|\tau^{1/2}\nabla\cdot\eta_T\|^2_{0,K} + \|\tau^{-1/2}\eta_u\|^2_{0,K} + 2\nu\|\varepsilon(\eta_u)\|^2_{0,K} + \|\delta^{1/2}\nabla\cdot\eta_u\|^2_{0K}$$
$$+ \|\tau^{1/2}(\nabla\eta_u)a\|^2_{0,K} + \|\delta^{-1/2}\eta_p\|^2_{0,K} + \|\tau^{1/2}\nabla\eta_p\|^2_{0,K}$$
$$\leq C((2\nu)^{-1}h_K^{2m+2}|T|^2_{m+1,K} + \sup_{x\in K}|a|_p h_K^{2k+1}|u|^2_{k+1,K} + \sup_{x\in K}|a|_p^{-1}h_K^{2l+1}|p|^2_{l+1,K})\,; \quad (27)$$

(b) *If* $0 \leq \mathrm{Re}_K(x) < 1$, $\forall x \in K$, *then*

$$\frac{1}{2\nu}\|\eta_T\|^2_{0,K} + \|\tau^{1/2}\nabla\cdot\eta_T\|^2_{0,K} + \|\tau^{-1/2}\eta_u\|^2_{0,K} + 2\nu\|\varepsilon(\eta_u)\|^2_{0,K} + \|\delta^{1/2}\nabla\cdot\eta_u\|^2_{0,K}$$
$$+ \|\tau^{1/2}(\nabla\eta_u)a\|^2_{0,K} + (2\nu)^{-1}\|\eta_p\|^2_{0,K} + \|\tau^{1/2}\nabla\eta_p\|^2_{0,K}$$
$$\leq C((2\nu)^{-1}h_K^{2m+2}|T|^2_{m+1,K} + 2\nu h_K^{2k}|u|^2_{k+1,K} + (2\nu)^{-1}h_K^{2l+2}|p|^2_{l+1,K})\,. \quad (28)$$

PROOF. The velocity and pressure estimates follow as in [9, 10]. For the stresses we have (a) Let $\mathrm{Re}_K(x) \geq 1$, $\forall x \in K$. Then

$$\frac{1}{2\nu}\|\eta_T\|^2_{0,K} + \|\tau^{1/2}\nabla\cdot\eta_T\|^2_{0,K} \leq \frac{1}{2\nu}(\|\eta_T\|^2_{0,K} + m_m h_K^2\|\nabla\cdot\eta_T\|^2_{0,K})$$
$$\leq \frac{C}{2\nu}h_K^{2m+2}|T|^2_{m+1,K}\,. \quad (29)$$

(b) Let $0 < \mathrm{Re}_K(x) \leq 1$, $\forall x \in K$. Then

$$\frac{1}{2\nu}\|\eta_T\|^2_{0,K} + \|\tau^{1/2}\nabla\cdot\eta_T\|^2_{0,K} = \frac{1}{2\nu}\left(\|\eta_T\|^2_{0,K} + \frac{m_m h_K^2}{4}\|\nabla\cdot\eta_T\|^2_{0,K}\right)$$
$$\leq \frac{C}{2\nu}h_K^{2m+2}|T|^2_{m+1,K}\,. \quad (30)$$

The last inequalities in (29), (30) follow by standard approximation theory [14]. □

PROOF OF THEOREM 3.1. Let $e_u^h = u_h - \tilde{u}_h$, $e_p^h = p_h - \tilde{p}_h$, $e_T^h = T_h - \tilde{T}_h$ and $e^u = e_u^h + \eta_u$, $e^p = e_p^h + \eta_p$, $e^T = e_T^h + \eta_T$. Then

$$\frac{1}{8\nu}\|e_T^h\|_0^2 + \alpha 2\nu\|\varepsilon(e_u^h)\|_0^2 + \frac{\alpha}{1+\alpha}\|\tau^{1/2}((\nabla e_u^h)a + \nabla e_p^h)\|_0^2 + \|\delta^{1/2}\nabla\cdot e_u^h\|_0^2$$
$$\leq B_1(e_T^h, e_p^h, e_u^h; e_T^h, e_p^h, -e_u^h) \quad \text{(by Lemma 3.1)}$$
$$= B_1(e^T - \eta_T, e^p - \eta_p, e^u - \eta_u; e_T^h, e_p^h, -e_u^h)$$
$$= -B_1(\eta_T, \eta_p, \eta_u; e_T^h, e_p^h, -e_u^h) \quad \text{(by consistency)}$$
$$= -\left\{\frac{1}{2\nu}(\eta_T, e_T^h) - (\varepsilon(\eta_u), e_T^h) + (\nabla\cdot\eta_u, e_p^h) + (\eta_T, \varepsilon(e_u^h))\right.$$

$$+ ((\nabla \boldsymbol{\eta}_u)\boldsymbol{a}, \boldsymbol{e}_u^h) - (\eta_p, \nabla \cdot \boldsymbol{e}_u^h) + (\nabla \cdot \boldsymbol{\eta}_u, \delta \nabla \cdot \boldsymbol{e}_u^h)$$

$$- \alpha 2\nu \left(\frac{1}{2\nu} \boldsymbol{\eta}_T - \boldsymbol{\varepsilon}(\boldsymbol{\eta}_u), \frac{1}{2\nu} \boldsymbol{e}_T^h + \boldsymbol{\varepsilon}(\boldsymbol{e}_u^h) \right)$$

$$+ \sum_{K \in \mathscr{C}_h} ((\nabla \boldsymbol{\eta}_u)\boldsymbol{a} + \nabla \eta_p - \nabla \cdot \boldsymbol{\eta}_T, \tau((\nabla \boldsymbol{e}_u^h)\boldsymbol{a} + \nabla e_p^h - \nabla \cdot \boldsymbol{e}_T^h))_K \Big\}$$

$$\leq \frac{\gamma_1}{4\nu} \|\boldsymbol{\eta}_T\|_0^2 + \frac{1}{4\nu\gamma_1} \|\boldsymbol{e}_T^h\|_0^2 + \gamma_1 \nu \|\boldsymbol{\varepsilon}(\boldsymbol{\eta}_u)\|_0^2 + \frac{1}{4\nu\gamma_1} \|\boldsymbol{e}_T^h\|_0^2$$

$$+ (\boldsymbol{\eta}_u, (\nabla \boldsymbol{e}_u^h)\boldsymbol{a} + \nabla e_p^h) + (\eta_p, \nabla \cdot \boldsymbol{e}_u^h) + \frac{\gamma_2}{\alpha 4\nu} \|\boldsymbol{\eta}_T\|_0^2 + \frac{\alpha 2\nu}{2\gamma_2} \|\boldsymbol{\varepsilon}(\boldsymbol{e}_u^h)\|_0^2$$

$$- (\nabla \cdot \boldsymbol{\eta}_u, \delta \nabla \cdot \boldsymbol{e}_u^h) + \frac{\alpha\gamma_1}{4\nu} \|\boldsymbol{\eta}_T\|_0^2 + \frac{\alpha}{4\nu\gamma_1} \|\boldsymbol{e}_T^h\|_0^2 + \alpha\gamma_1 \nu \|\boldsymbol{\varepsilon}(\boldsymbol{\eta}_u)\|_0^2 + \frac{\alpha}{4\nu\gamma_1} \|\boldsymbol{e}_T^h\|_0^2$$

$$+ \frac{\alpha\gamma_2}{4\nu} \|\boldsymbol{\eta}_T\|_0^2 + \frac{\alpha\nu}{\gamma_2} \|\boldsymbol{\varepsilon}(\boldsymbol{e}_u^h)\|_0^2 + \alpha\gamma_2 \nu \|\boldsymbol{\varepsilon}(\boldsymbol{\eta}_u)\|_0^2 + \frac{\alpha\nu}{\gamma_2} \|\boldsymbol{\varepsilon}(\boldsymbol{e}_u^h)\|_0^2$$

$$+ \frac{\gamma_3}{2} \|\tau^{1/2}(\nabla \boldsymbol{\eta}_u)\boldsymbol{a}\|_0^2 + \frac{1}{2\gamma_3} \|\tau^{1/2}((\nabla \boldsymbol{e}_u^h)\boldsymbol{a} + \nabla e_p^h)\|_0^2 + \frac{\gamma_3}{2} \|\tau^{1/2} \nabla \eta_p\|_0^2$$

$$+ \frac{1}{2\gamma_3} \|\tau^{1/2}((\nabla \boldsymbol{e}_u^h)\boldsymbol{a} + \nabla e_p^h)\|_0^2 + \sum_{K \in \mathscr{C}_h} \frac{\gamma_3}{2} \|\tau^{1/2} \nabla \cdot \boldsymbol{\eta}_T\|_{0,K}^2$$

$$+ \frac{1}{2\gamma_3} \|\tau^{1/2}((\nabla \boldsymbol{e}_u^h)\boldsymbol{a} + \nabla e_p^h)\|_0^2$$

$$+ \frac{\gamma_1}{2} \|\tau^{1/2}(\nabla \boldsymbol{\eta}_u)\boldsymbol{a}\|_0^2 + \frac{1}{2\gamma_1} \sum_{K \in \mathscr{C}_h} \|\tau^{1/2} \nabla \cdot \boldsymbol{e}_T^h\|_{0,K}^2 + \frac{1}{2\gamma_1} \sum_{K \in \mathscr{C}_h} \|\tau^{1/2} \nabla \cdot \boldsymbol{e}_T^h\|_{0,K}^2$$

$$+ \frac{\gamma_1}{2} \|\tau^{1/2} \nabla \eta_p\|_0^2 + \sum_{K \in \mathscr{C}_h} \frac{\gamma_1}{2} \|\tau^{1/2} \nabla \cdot \boldsymbol{\eta}_T\|_{0,K}^2 + \frac{1}{2\gamma_1} \sum_{K \in \mathscr{C}_h} \|\tau^{1/2} \nabla \cdot \boldsymbol{e}_T^h\|_{0,K}^2$$

$$\leq \frac{1}{4\nu} \left(\gamma_1 + \frac{\gamma_2}{\alpha} + \alpha\gamma_1 + \alpha\gamma_2 \right) \|\boldsymbol{\eta}_T\|_0^2 + \frac{1}{4\nu} \left(\frac{2}{\gamma_1} + \frac{2\alpha}{\gamma_1} \right) \|\boldsymbol{e}_T^h\|_0^2$$

$$+ \nu(\gamma_1 + \alpha\gamma_1 + \alpha\gamma_2) \|\boldsymbol{\varepsilon}(\boldsymbol{\eta}_u)\|_0^2 + \frac{\gamma_3}{2} \|\tau^{-1/2} \boldsymbol{\eta}_u\|_0^2 + \frac{1}{2\gamma_3} \|\tau^{1/2}((\nabla \boldsymbol{e}_u^h)\boldsymbol{a} + \nabla e_p^h)\|_0^2$$

$$+ (\eta_p, \nabla \cdot \boldsymbol{e}_u^h) + \frac{3\nu\alpha}{\gamma_2} \|\boldsymbol{\varepsilon}(\boldsymbol{e}_u^h)\|_0^2$$

$$+ \frac{1}{2\gamma_4} \|\delta^{1/2} \nabla \cdot \boldsymbol{e}_u^h\|_0^2 + \frac{\gamma_4}{2} \|\delta^{1/2} \nabla \cdot \boldsymbol{\eta}_u\|_0^2 + \frac{\gamma_1 + \gamma_3}{2} \|\tau^{1/2}(\nabla \boldsymbol{\eta}_u)\boldsymbol{a}\|_0^2$$

$$+ \frac{3}{2\gamma_3} \|\tau^{1/2}((\nabla \boldsymbol{e}_u^h)\boldsymbol{a} + \nabla e_p^h)\|_0^2 + \frac{\gamma_1 + \gamma_3}{2} \|\tau^{1/2} \nabla \eta_p\|_0^2$$

$$+ \sum_{K \in \mathscr{C}_h} \frac{\gamma_1 + \gamma_3}{2} \|\tau^{1/2} \nabla \cdot \boldsymbol{\eta}_T\|_{0,K}^2 + \frac{3}{2\gamma_1} \sum_{K \in \mathscr{C}_h} \|\tau^{1/2} \nabla \cdot \boldsymbol{e}_T^h\|_{0,K}^2$$

$$
\begin{aligned}
&\leqslant \frac{1}{4\nu}\left(\frac{7}{2\gamma_1}+\frac{2\alpha}{\gamma_1}\right)\|e_T^h\|_0^2+\frac{3\nu\alpha}{\gamma_2}\|\varepsilon(e_u^h)\|_0^2 \\
&+\frac{2}{\gamma_3}\|\tau^{1/2}((\nabla e_u^h)a+\nabla e_p^h)\|_0^2+\frac{1}{2\gamma_4}\|\delta^{1/2}\nabla\cdot e_u^h\|_0^2+(\eta_p,\nabla\cdot e_u^h) \\
&+\frac{1}{4\nu}\left((1+\alpha)\gamma_1+\left(\frac{1}{\alpha}+\alpha\right)\gamma_2\right)\|\eta_T\|_0^2+\sum_{K\in\mathscr{C}_h}\frac{\gamma_1+\gamma_3}{2}\|\tau^{1/2}\nabla\cdot\eta_T\|_{0,K}^2 \\
&+\frac{\gamma_3}{2}\|\tau^{-1/2}\eta_u\|_0^2+\nu((1+\alpha)\gamma_1+\alpha\gamma_2)\|\varepsilon(\eta_u)\|_0^2+\frac{\gamma_4}{2}\|\delta^{1/2}\nabla\cdot\eta_u\|_0^2 \\
&+\frac{\gamma_1+\gamma_3}{2}\|\tau^{1/2}(\nabla\eta_u)a\|_0^2+\frac{\gamma_1+\gamma_3}{2}\|\tau^{1/2}\nabla\eta_p\|_0^2\,.
\end{aligned}
$$

Depending on whether the element is advectively or diffusively dominated, we have control over $\nabla\cdot e_u^h$ in a different manner, namely

$$
\begin{aligned}
(\eta_p,\nabla\cdot e_u^h)\leqslant \sum_{K\in\mathscr{C}_h}\Bigg[&\frac{\gamma_4}{2}\,\mathrm{H}(\mathrm{Re}_K-1)\|\delta^{-1/2}\eta_p\|_{0,K}^2+\frac{1}{2\gamma_4}\,\mathrm{H}(\mathrm{Re}_K-1)\|\delta^{1/2}\nabla\cdot e_u^h\|_{0,K}^2 \\
&+\frac{\gamma_2}{2\alpha}\,\mathrm{H}(1-\mathrm{Re}_K)(2\nu)^{-1}\|\eta_p\|_{0,K}^2+\frac{1}{2\gamma_2}\,\mathrm{H}(1-\mathrm{Re}_K)\alpha 2\nu\|\nabla\cdot e_u^h\|_{0,K}^2\Bigg] \\
\leqslant \sum_{K\in\mathscr{C}_h}\Bigg[&\frac{1}{2\gamma_4}\,\mathrm{H}(\mathrm{Re}_K-1)\|\delta^{1/2}\nabla\cdot e_u^h\|_{0,K}^2+\frac{N}{2\gamma_2}\,\alpha\mathrm{H}(1-\mathrm{Re}_K)2\nu\|\varepsilon(e_u^h)\|_{0,K}^2 \\
&+\frac{\gamma_4}{2}\,\mathrm{H}(\mathrm{Re}_K-1)\|\delta^{-1/2}\eta_p\|_{0,K}^2+\frac{\gamma_2}{2\alpha}\,\mathrm{H}(1-\mathrm{Re}_K)(2\nu)^{-1}\|\eta_p\|_{0,K}^2\Bigg]\,.
\end{aligned}
$$

Combining with the previous estimate, and selecting $\gamma_1=2(7+4\alpha)$, $\gamma_2=3+N\mathrm{H}(1-\mathrm{Re}_K)$, $\gamma_3=4(1+\alpha)/\alpha$ and $\gamma_4=1+\mathrm{H}(\mathrm{Re}_K-1)$, gives

$$
\begin{aligned}
&\frac{1}{16\nu}\|e_T^h\|_0^2+\alpha\nu\|\varepsilon(e_u^h)\|_0^2+\frac{\alpha}{2(1+\alpha)}\|\tau^{1/2}((\nabla e_u^h)a+\nabla e_p^h)\|_0^2+\frac{1}{2}\|\delta^{1/2}\nabla\cdot e_u^h\|_0^2 \\
&\leqslant \sum_{K\in\mathscr{C}_h}\Bigg[\frac{1}{4\nu}\left(2(1+\alpha)(7+4\alpha)+\left(\frac{1}{\alpha}+\alpha\right)(3+N\mathrm{H}(1-\mathrm{Re}_K))\right)\|\eta_T\|_{0,K}^2 \\
&\quad+\left((7+4\alpha)+\frac{2(1+\alpha)}{\alpha}\right)\|\tau^{1/2}\nabla\cdot\eta_T\|_{0,K}^2 \\
&\quad+\frac{2(1+\alpha)}{\alpha}\|\tau^{-1/2}\eta_u\|_{0,K}^2+\frac{1+\mathrm{H}(\mathrm{Re}_K-1)}{2}\|\delta^{1/2}\nabla\cdot\eta_u\|_{0,K}^2 \\
&\quad+\nu((1+\alpha)2(7+4\alpha)+\alpha(3+N\mathrm{H}(1-\mathrm{Re}_K)))\|\varepsilon(\eta_u)\|_{0,K}^2 \\
&\quad+\left(7+4\alpha+\frac{2(1+\alpha)}{\alpha}\right)\|\tau^{1/2}(\nabla\eta_u)a\|_{0,K}^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(7+4\alpha+\frac{2(1+\alpha)}{\alpha}\right)\|\tau^{1/2}\nabla\eta_p\|_{0,K}^2\\
&+\frac{1+\mathrm{H}(\mathrm{Re}_K-1)}{2}\,\mathrm{H}(\mathrm{Re}_K-1)\|\delta^{-1/2}\eta_p\|_{0,K}^2\\
&+\left.\frac{3+N\mathrm{H}(1-\mathrm{Re}_K)}{2\alpha}\,\mathrm{H}(1-\mathrm{Re}_K)(2\nu)^{-1}\|\eta_p\|_{0,K}^2\right]\\
\leqslant C(\alpha)\Big(&\frac{1}{2\nu}\|\boldsymbol{\eta}_T\|_0^2+\|\tau^{1/2}\nabla\cdot\boldsymbol{\eta}_T\|_0^2+\|\tau^{-1/2}\boldsymbol{\eta}_u\|_0^2+2\nu\|\boldsymbol{\varepsilon}(\boldsymbol{\eta}_u)\|_0^2+\|\delta^{1/2}\nabla\cdot\boldsymbol{\eta}_u\|_0^2\\
&+\|\tau^{1/2}(\nabla\boldsymbol{\eta}_u)\boldsymbol{a}\|_0^2+\|\tau^{1/2}\nabla\eta_p\|_0^2\\
&+\sum_{K\in\mathscr{C}_h}\left[\mathrm{H}(\mathrm{Re}_K-1)\|\delta^{-1/2}\eta_p\|_{0,K}^2+\mathrm{H}(1-\mathrm{Re}_K)(2\nu)^{-1}\|\eta_p\|_{0,K}^2\right]\Big)\\
\leqslant C(\alpha)\sum_{K\in\mathscr{C}_h}\Big(&(2\nu)^{-1}h_K^{2m+2}|\boldsymbol{T}|_{m+1,K}^2\\
&+h_K^{2k}|\boldsymbol{u}|_{k+1,K}^2(\mathrm{H}(\mathrm{Re}_K-1)h_K\sup_{x\in K}|\boldsymbol{a}|_p+\mathrm{H}(1-\mathrm{Re}_K)2\nu)\\
&+h_K^{2l}|p|_{l+1,K}^2(\mathrm{H}(\mathrm{Re}_K-1)h_K\sup_{x\in K}|\boldsymbol{a}|_p^{-1}+\mathrm{H}(1-\mathrm{Re}_K)h_K^2(2\nu)^{-1})\Big)\,.
\end{aligned}
$$

The last inequality follows by the interpolation estimate (Lemma 3.2). The result follows using the interpolation estimate for the interpolation errors in the norm on the left-hand side and then applying the triangle inequality. □

REMARK 4. The interested reader can establish a convergence analysis for Method II in the same manner as shown above for Method I. The only significant change is in the stability argument for Method II for which any $\alpha > 0$ renders the methodology stable. The stability arguments follow similar lines as in [10].

4. The incompressible Navier–Stokes equations

The steady-state incompressible Navier–Stokes equations in the velocity–pressure–extra stress formulation can be written as follows:

$$
\begin{aligned}
&\frac{1}{2\nu}\boldsymbol{T}-\boldsymbol{\varepsilon}(\boldsymbol{u})=\boldsymbol{0}\quad\text{in }\Omega\,,\qquad \nabla\cdot\boldsymbol{u}=0\quad\text{in }\Omega\,,\\
&(\nabla\boldsymbol{u})\boldsymbol{u}-\nabla\cdot\boldsymbol{T}+\nabla p=\boldsymbol{f}\quad\text{in }\Omega\,,\\
&\boldsymbol{u}=\boldsymbol{g}\quad\text{on }\Gamma_g\,,\qquad(\boldsymbol{T}-p\boldsymbol{I})\boldsymbol{n}=\boldsymbol{h}\quad\text{on }\Gamma_h\,,
\end{aligned}
\tag{31}
$$

where in addition to the notation introduced in Section 2, $\boldsymbol{g}$ and $\boldsymbol{h}$ are given Dirichlet and Neumann boundary conditions, respectively, specified on Γ_g and Γ_h, complementary subsets of Γ. Let us define the set of trial functions for velocity as

$$V_h^g = \{v \in H^1(\Omega)^N \mid v|_{x\in K} \in R_k(K)^N,\ K \in \mathscr{C}_h,\ v = g \text{ on } \Gamma_g\}\ . \tag{32}$$

Employing (2)–(4) and (32), the first stabilized method introduced in Section 2 can be extended as follows: Find $u_h \in V_h^g$, $p_h \in P_h$ and $T_h \in W_h$ such that

$$B(T_h, p_h, u_h; S, q, v) = F(S, q, v)\ , \quad (S, q, v) \in W_h \times P_h \times V_h\ , \tag{33}$$

with

$$\begin{aligned} B(T, p, u; S, q, v) &= \frac{1}{2\nu}(T, S) - (\varepsilon(u), S) + (\nabla\cdot u, q) - (T, \varepsilon(v)) - ((\nabla u)u, v) \\ &+ (p, \nabla\cdot v) - (\nabla\cdot u, \delta\nabla\cdot v) - \alpha 2\nu\Big(\frac{1}{2\nu}T - \varepsilon(u), \frac{1}{2\nu}S - \varepsilon(v)\Big) \\ &+ \sum_{K\in\mathscr{C}_h} ((\nabla u)u + \nabla p - \nabla\cdot T, \tau(-(\nabla v)u + \nabla q - \nabla\cdot S))_K \end{aligned} \tag{34}$$

and

$$F(S, q, v) = -(f, v) - (h, v)_{\Gamma_h} + \sum_{K\in\mathscr{C}_h} (f, \tau(-(\nabla v)u + \nabla q - \nabla\cdot S))_K\ , \tag{35}$$

where the stability parameters δ, τ and α are defined as follows:

$$\delta = \lambda|u(x)|_p h_K \xi(\mathrm{Re}_K(x))\ , \tag{36}$$

$$\tau(x, \mathrm{Re}_K(x)) = \frac{h_K}{2|u(x)|_p}\xi(\mathrm{Re}_K(x))\ , \tag{37}$$

$$0 < \alpha \leq 0.5\ , \tag{38}$$

$$\mathrm{Re}_K(x) = \frac{m_m|u(x)|_p h_K}{4\nu(x)}\ , \tag{39}$$

$$\xi(\mathrm{Re}_K(x)) = \begin{cases} \mathrm{Re}_K(x)\ , & 0 \leq \mathrm{Re}_K(x) < 1\ , \\ 1\ , & \mathrm{Re}_K(x) \geq 1\ , \end{cases} \tag{40}$$

$$|u(x)|_p = \begin{cases} (\Sigma_{i=1}^N |u_i(x)|^p)^{1/p}\ , & 1 \leq p < \infty\ , \\ \max_{i=1,N} |u_i(x)|\ , & p = \infty\ , \end{cases} \tag{41}$$

$$m_m = \min\Big\{\frac{1}{3}, 2C_m\Big\}\ , \tag{42}$$

$$C_m \sum_{K\in\mathscr{C}_h} h_K^2 \|\nabla\cdot T\|_{0,K}^2 \leq \|T\|_0^2\ , \quad T \in W_h\ , \tag{43}$$

and λ is a positive parameter.

5. Numerical tests

We have performed numerical experiments for this method employing equal-order bilinear approximations for all variables (T, p, u). The constant $m_{m=1}$ is set to 1/3 throughout.

The first test problem is the 'leaky' cavity consisting of the flow in a unit square domain with boundary conditions $u_1 = 1$, $u_2 = 0$ at $y = 1$ $(0 \leq x \leq 1)$ and $\boldsymbol{u} = \boldsymbol{0}$ at all other boundaries. A uniform mesh of 32×32 elements is used. In Fig. 1 we plot contours of each extra stress component and of pressure, the velocity vectors and the stream function contours at Re = 400, for $\alpha = 0.1$. We also tested the method for $\alpha = 0.02$ and 0 and found the results to be very similar. The case $\alpha = 0$ is not covered in our convergence analysis, but yields just as good results. We repeated this study for the same set of values of α at Re = 5000, and plotted the

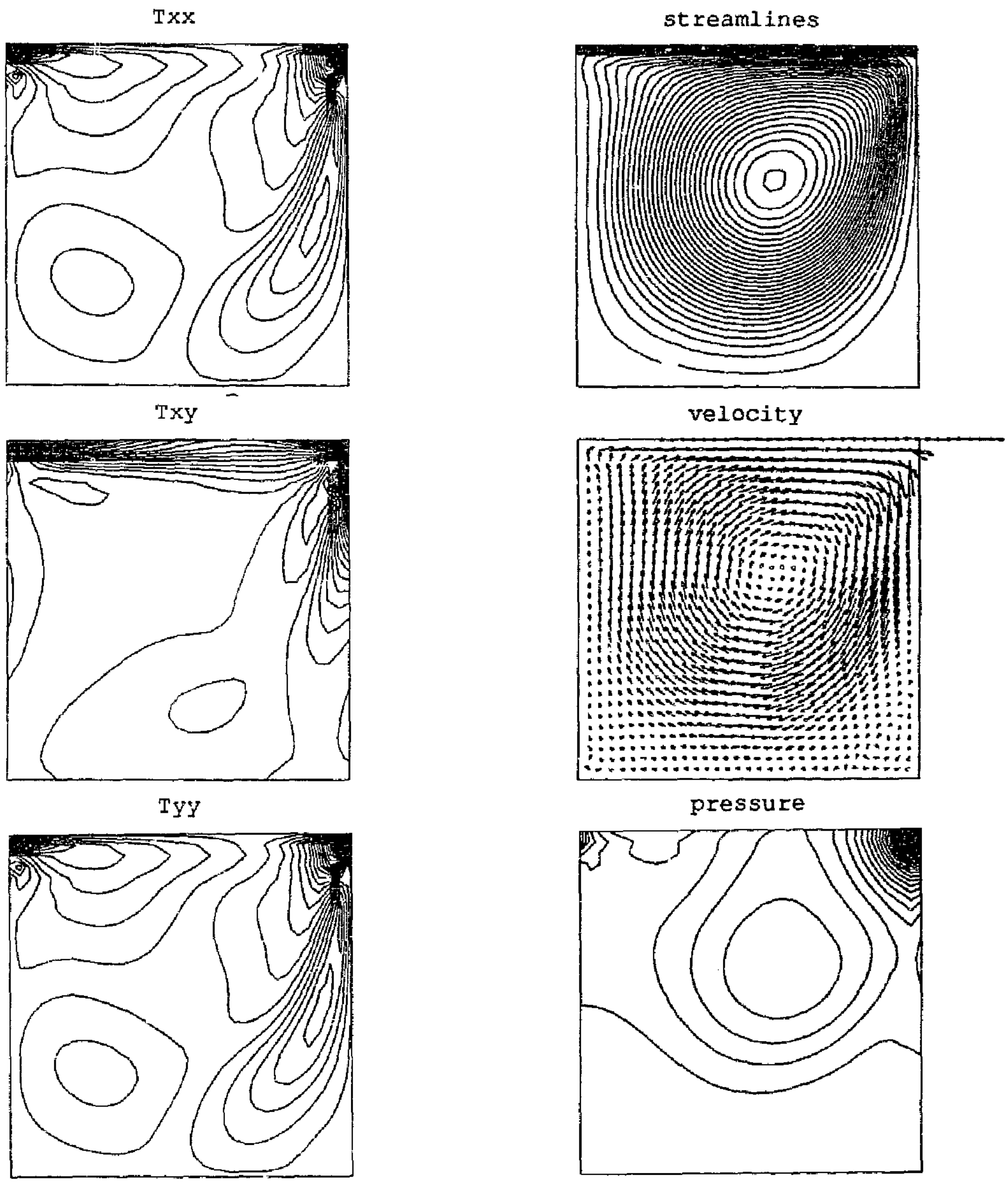

Fig. 1. Extra stress, stream function and pressure contours, and velocity vectors for the 'leaky' cavity flow at Re = 400: 32×32 Q1/Q1/Q1 elements with $\alpha = 0.1$.

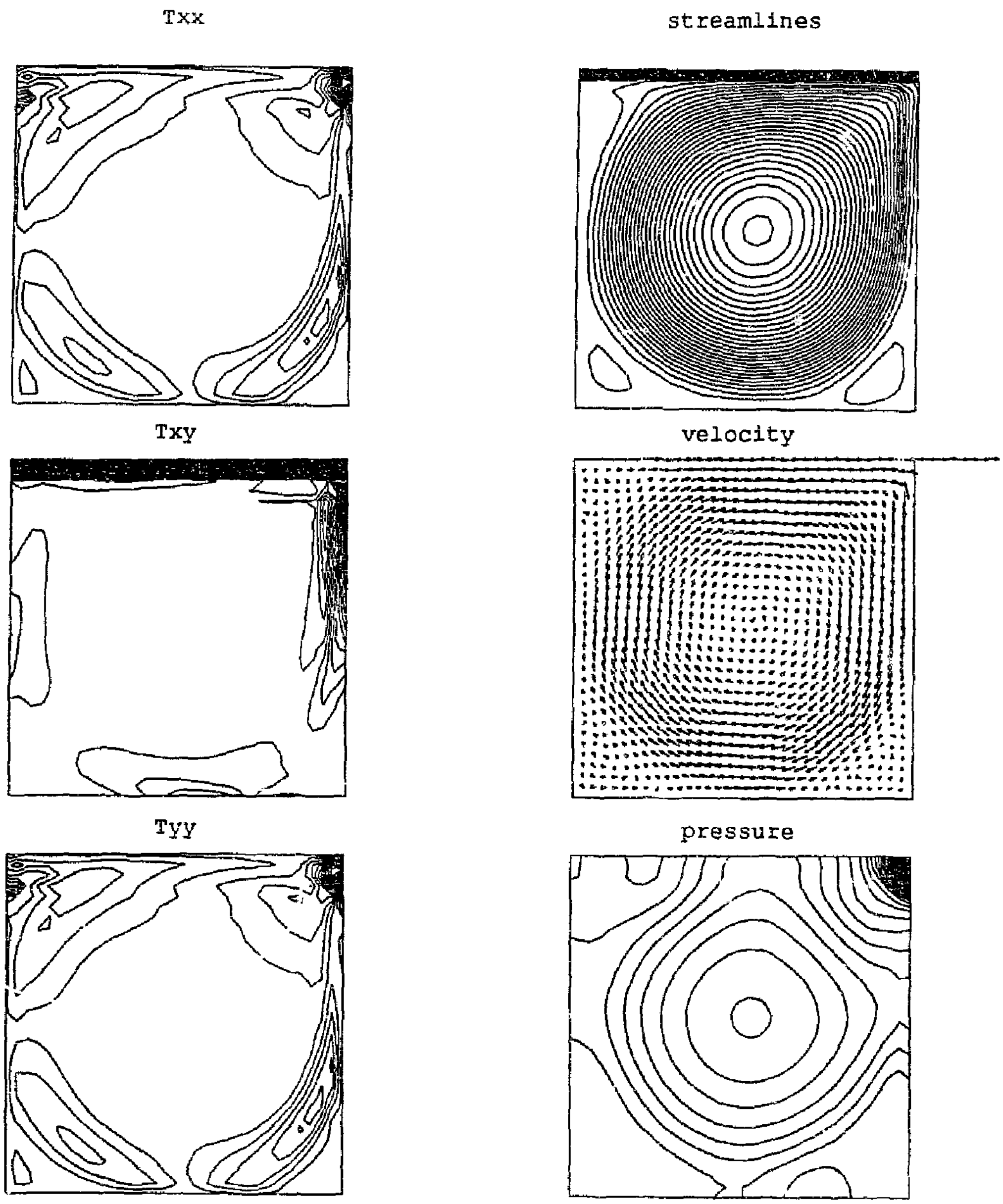

Fig. 2. Extra stress, stream function and pressure contours, and velocity vectors for the 'leaky' cavity flow at Re = 5000: 32 × 32 Q1/Q1/Q1 elements with $\alpha = 0.1$.

results, again for $\alpha = 0.1$, in Fig. 2; the method gives results comparable to those in [10, 15]. In Figs. 3 and 4 we compare the present results for $\alpha = 0.02$, with the results obtained with the equal-order bilinear velocity–pressure formulation using the same definitions for τ and δ in (36) and (37), at Re = 400 and 5000.

Next, the steady flow past a cylinder is studied at Re = 26 with the mesh (1965 elements) shown in Fig. 5, for $\alpha = 0.1$, 0.02 and 0. In Fig. 6 we show the results for $\alpha = 0.1$. The method

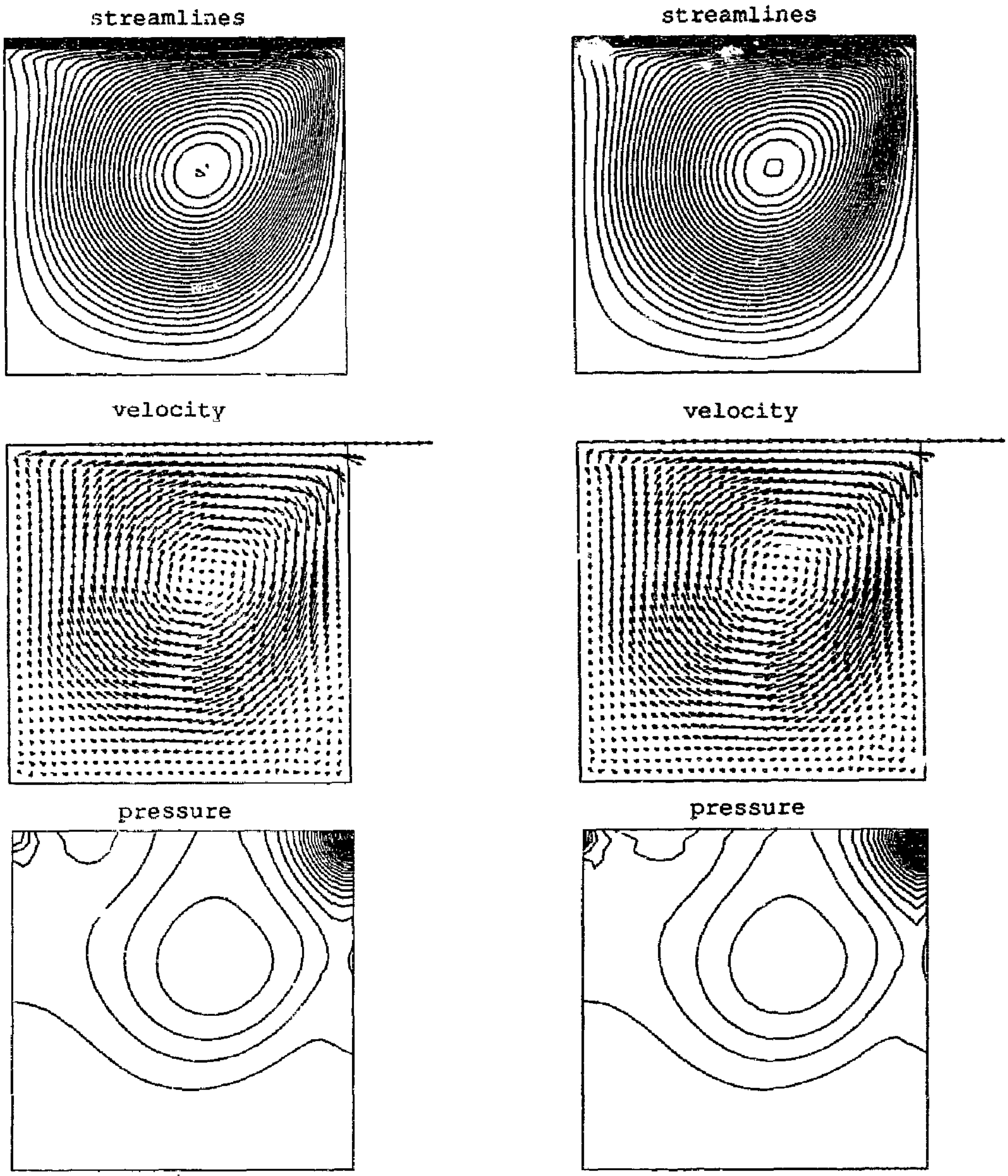

Fig. 3. Comparison between the equal-order bilinear velocity–pressure formulation (left) and the present velocity–pressure–stress formulation (right) at Re = 400.

gives almost the same results as for $\alpha = 0.02$ and 0. We note that these results compare qualitatively well with experimental data (see [16]).

Acknowledgment

During the course of this work, Leopoldo P. Franca has been partially supported by the

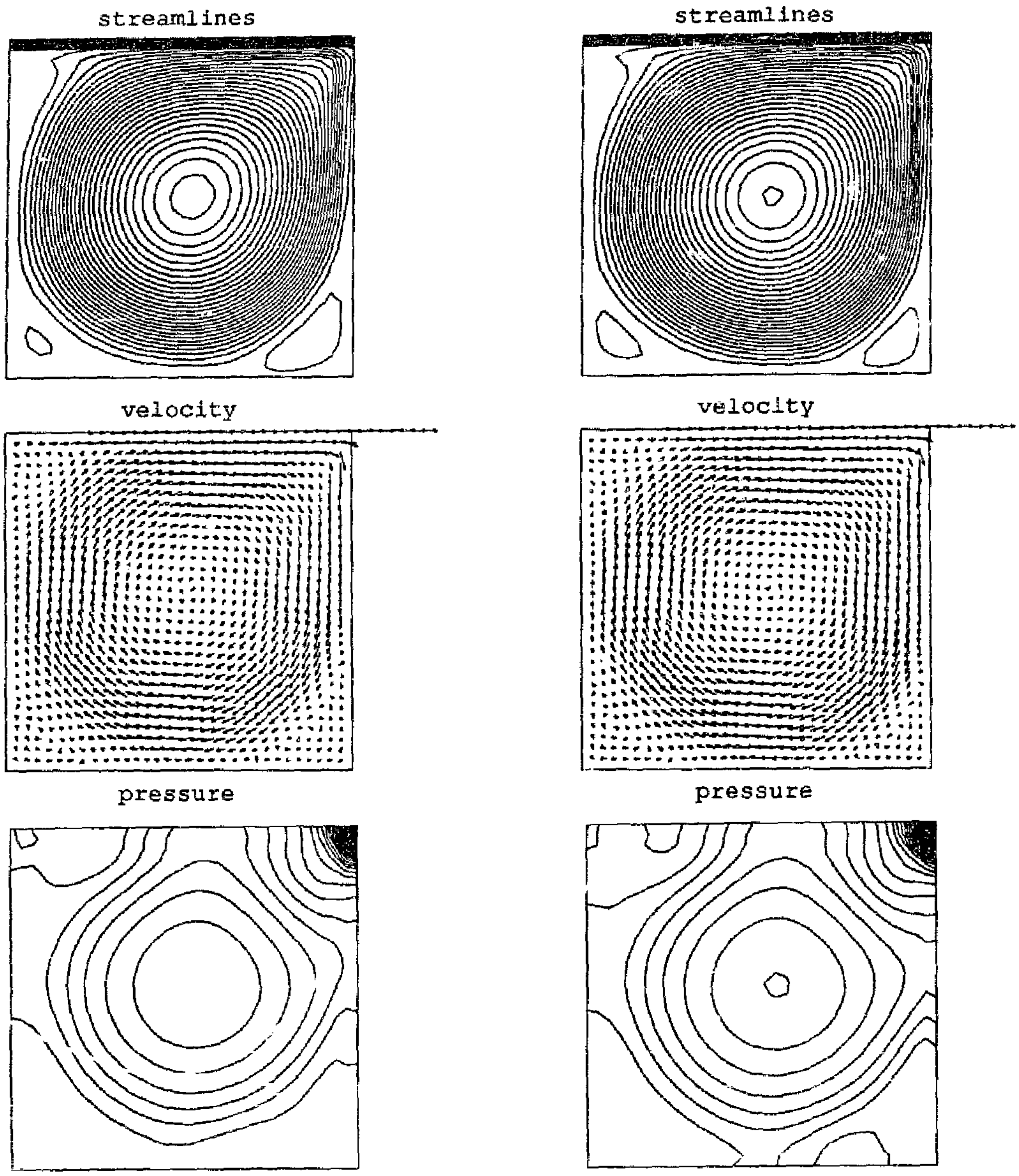

Fig. 4. Comparison between the equal-order bilinear velocity-pressure formulation (left) and the present velocity-pressure-stress formulation (right) at Re = 5000.

Army High Performance Computing Research Center and by CNPq Procs. 300419/90-2 and 402598/90-3 and Tayfun E. Tezduyar has been partially supported by CNPq Proc. 453238/91-2. This research was sponsored by NASA-Johnson Space Center under grant NAG 9-449 and by NSF under grant MSM-8796352.

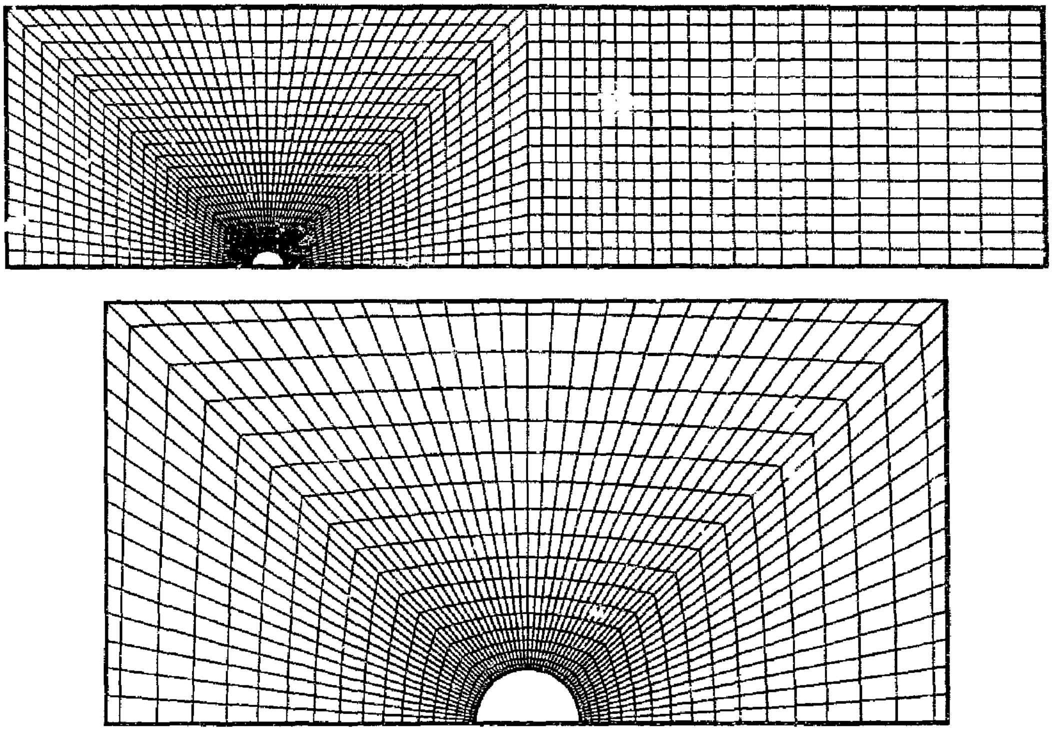

Fig. 5. Mesh (1965 elements) employed for the cylinder problem.

References

[1] J. Baranger and K. Najib, Analyse numérique des écoulements quasi-newtoniens dont la viscosité obéit à la loi puissance ou la loi de Carreau, Numer. Math. 58 (1990) 35–49.

[2] J. Baranger and D. Sandri, A formulation of Stoke's problem and the linear elasticity equations suggested by Oldroyd model for viscoelastic flow, M^2AN 26 (1992) 331–345.

[3] J. Baranger and D. Sandri, Finite element approximation of viscoelastic fluid flow: Existence of approximate solutions and error bounds. I. Discontinuous constraints, Report no. 115, Equipe d'Analyse Numérique Lyon-Saint-Etienne, 1991.

[4] M. Fortin and D. Esselaoui, A finite element procedure for viscoelastic flows, in: R.H. Gallagher, R. Glowinski, P.M. Gresho, J.T. Oden and O.C. Zienkiewicz, eds., Finite Elements in Fluids, Vol. 7 (Wiley, New York, 1987) 47–64.

[5] M. Fortin and A. Fortin, A new approach for the FEM simulation of viscoelastic flows, J. Non-Newtonian Fluid Mech. 32 (1989) 295–310.

[6] M. Fortin and R. Pierre, On the convergence of the mixed method of Crochet and Marchal for viscoelastic flows, Comput. Methods Appl. Mech. Engrg. 73 (1989) 341–350.

[7] J.M. Marchal and M.J. Crochet, A new mixed finite element for calculating viscoelastic flow, J. Non-Newtonian Fluid Mech. 26 (1987) 77–114.

[8] L.P. Franca and R. Stenberg, Error analysis of some Galerkin-least-squares methods for the elasticity equations, SIAM J. Numer. Anal. 26 (1991) 1680–1697.

[9] L.P. Franca, S.L. Frey and T.J.R. Hughes, Stabilized finite element methods: I. Application to the advective-diffusive model, Comput. Methods Appl. Mech. Engrg. 95 (1992) 253–276.

[10] L.P. Franca and S.L. Frey, Stabilized finite element methods: II. The incompressible Navier–Stokes equations, Comput. Methods Appl. Mech. Engrg. 99 (1992) 209–233.

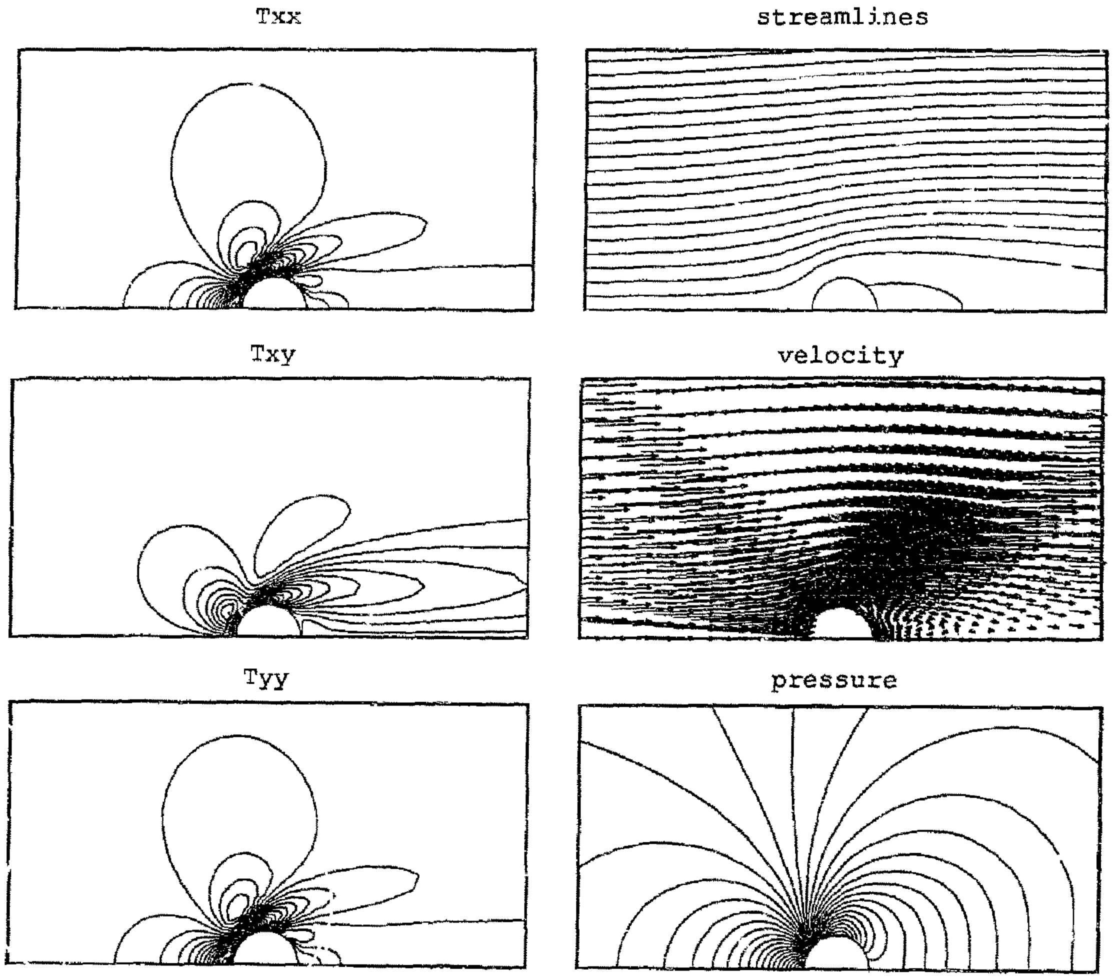

Fig. 6. Flow past a cylinder at Re = 26 with $\alpha = 0.1$.

[11] T.E. Tezduyar, M. Behr and J. Liou, A new strategy for finite element computations involving moving boundaries and interfaces—The deforming-spatial-domain/space-time procedure: I. The concept and the preliminary numerical tests, Comput. Methods Appl. Mech. Engrg. 94 (1992) 339–351.

[12] T.E. Tezduyar, M. Behr, S. Mittal and J. Liou, A new strategy for finite element computations involving moving boundaries and interfaces—The deforming-spatial-domain/space-time procedure: II. Computation of free-surface flows, two-liquid flows, and flows with drifting cylinders, Comput. Methods Appl. Mech. Engrg. 94 (1992) 353–371.

[13] T.E. Tezduyar, S. Mittal, S.E. Ray and R. Shih, Incompressible flow computations with stabilized bilinear and linear equal-order-interpolation velocity-pressure elements, Comput. Methods Appl. Mech. Engrg. 95 (1992) 221–242.

[14] P.G. Ciarlet, Basic error estimates for elliptic problems, in: P.G. Ciarlet and J.L. Lions eds., Handbook of Numerical Analysis, Vol. II, Finite Element Methods (Part 1) (North-Holland, Amsterdam, 1991) 17–351.

[15] P.M. Gresho and S.T. Chan, On the theory of semi-implicit projection methods for viscous incompressible flow and its implementation via a finite element method that also introduces a nearly consistent mass matrix. Part 2: Implementation, Internat. J. Numer. Methods Fluids 11 (1990) 621–659.

[16] M. Van Dyke, An Album of Fluid Motion (The Parabolic Press, 1982).